Quantum gravity with a positive cosmological constant

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ABSTRACT

A quantum theory of gravity is described in the case of a positive cosmological constant in $3+1$ dimensions. Both old and new results are described, which support the case that loop quantum gravity provides a satisfactory quantum theory of gravity. These include the existence of a ground state, discoverd by Kodama, which both is an exact solution to the constraints of quantum gravity and has a semiclassical limit which is deSitter spacetime. The long wavelength excitations of this state are studied and are shown to reproduce both gravitons and, when matter is included, quantum field theory on deSitter spacetime. Furthermore, one may derive directly from the Wheeler-deWitt equation corrections to the energy-momentum relations for matter fields of the form $E^2 = p^2 + m^2 + \alpha l_{Pl} E^3 + \ldots$ where α is a computable dimensionless constant. This may lead in the next few years to experimental tests of the theory.

To study the excitations of the Kodama state exactly requires the use of the spin network representation, which is quantum deformed due to the cosmological constant. The theory may be developed within a single horizon, and the boundary states described exactly in terms of a boundary Chern-Simons theory. The Bekenstein bound is recovered and the N bound of Banks is given a background independent explanation.

The paper is written as an introduction to loop quantum gravity, requiring no prior knowledge of the subject. The deep relationship between quantum gravity and topological field theory is stressed throughout.

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1 Introduction

Of the different approaches to quantum gravity, the most conservative is likely loop quantum gravity[1, 2, 3]. This approach is based, to begin with, on the quantization of Einstein's theory of general relativity, using a particular formulation discovered by Sen[4] and completed by Ashtekar[5]. Rather than postulating new degrees of freedom, symmetries or dimensions, loop quantum gravity takes the basic principles of general relativity and quantum field theory seriously, and puts the emphasis on the development of methods that do not compromise either set of principles. These methods highlight the force that the principles of relativity theory, primarily diffeomorphism invariance and the independence from any fixed, non-dynamical background structure have, when treated properly in the context of quantum field theory. Indeed, it turns out that once this is done, the theory admits a wide range of assumptions concerning the fundamental degrees of freedom, symmetries and supersymmetries, as well as the exact dynamical laws. Einstein's equations may be imposed, and to a remarkable extent, solved, quantum mechanically, but other assumptions concerning the fundamental dynamics may also be studied.

Loop quantum gravity has been under development since 1986, and throughout this time there has been continual progress. The various obstacles encountered have in most cases been overcome. As a result it has been possible over time to make increasingly strong claims for this approach to quantum gravity.

This paper is concerned with a subset of the results, those relevant for the case that the cosmological constant is non-zero and positive. For, in this particular case, there are now, as I will describe below, sufficient results to claim that the theory fully deserves the status of a candidate for the theory of quantum spacetime.

Some of the key results on which this claim is based are old, others are more recent. The claim that loop quantum gravity with $\Lambda > 0$ provides a consistent quantum theory of gravity includes the following:

Results special for $\Lambda > 0$.

- The existence of an exact physical state, which exactly solves all the quantum constraint equations that define the theory, which also has a semiclassical interpretation, as a WKB state for deSitter spacetime. This is the Kodama state[6], which is described in section 7 below. The existence of this state solves a major problem faced by loop quantum gravity, as it shows that, at least for $\Lambda \neq 0$, the theory does have a good low energy limit which reproduces general relativity and quantum field theory.

- Small, long wavelength perturbations of the Kodama state do, for small
$$\lambda = \Lambda G\hbar, \tag{1}$$
reproduce the spectrum of gravitons on a background of deSitter spacetime. (Section 9)

- The theory may be coupled to arbitrary matter fields. For the same conditions, long wavelength and small λ, perturbations of the Kodama state in the matter sector reproduce QFT on the deSitter background[7]. (Section 8)

- When one extends the approximation to higher order terms in $l_{Pl}E$ one finds corrections to the energy-momentum relations for matter fields, of a form,
$$E^2 = p^2 + m^2 + \alpha E^3 + \dots \tag{2}$$

Such corrections are in fact amenable to experimental test[9, 10, 11, 12] in present and near future experiments (section 10).

- There is a natural boundary term[13] which may be added to study the case of horizons[14, 15] or timelike boundaries[16, 17]. This leads to an explicit construction of the boundary Hilbert space. One consequence is that the Bekenstein bound is satisfied automatically[13]. Another is a new understanding, in purely background independent terms, of the N bound conjectured by Banks[18, 19].

To these may be added:

Results of loop quantum gravity for all Λ

- There is a detailed Hamiltonian quantization including states, inner product, observables, regularization procedures etc[1, 2, 20, 21, 22]. The space of diffeomorphism invariant states is known precisely and is characterized in terms of spin networks[23], which provide an exact orthonormal basis[24].

- Among the operators which are understood, and which are finite after a regularization procedure[3, 25], are the area of surfaces (such as the boundary, when one is present) and the volume of regions, including the volume of the universe. These have known, discrete spectra, leading to a picture of discrete spatial geometry[26, 27]. Another operator which is finite and well defined is the hamiltonian constraint, as well as the hamiltonian in certain fixed gauges or boundary conditions[28, 29, 30].

- A path integral formulation is known, called spin foams, for computing amplitudes for evolution of spin network states[32, 33, 34, 35, 36, 37, 38]. This may be derived in several different ways, whose agreement stands as evidence for the robustness of the formulation. The relation to the Hamiltonian theory is also understood, in the conventional fashion[33].

- These results explain the failure of perturbative approaches to quantum general relativity and supergravity, as those assume the existence of states of the full theory that correspond to graviton states in the linearized theory of arbitrarily short wavelength. Such states do not exist in loop quantum gravity, and it is fully understood why, and why this is not an obstacle to the theory being consistent.

- There is a general theory of boundaries, which includes horizons[13, 14, 15]. This leads to a complete description of black hole and cosmological horizons, reproduces the Bekenstein bound and gives an explanation of the Bekenstein-Hawking entropy in terms of microstates per microstate, where the macrostate is the classical description of an horizon and the microstates are the exact states of the quantum geometry description coming from loop quantum gravity. It is also possible to deduce from this corrections to the Hawking radiation[39], in particular, both logarithmic corrections to the entropy formula[40] and a fine structure to the hawking radiation[41, 39].

- There is a general approach to cosmological models, in which the full quantum theory is reduced to homogeneous spacetimes, called *loop quantum cosmology*[42]. The results reproduce those of semiclassical cosmology for times large in Planck units. However the cosmological singularity is removed, and a bounce is predicted[42].

- Many of these results are confirmed by rigorous results and theorems, at the level of mathematical quantum field theory[43, 44, 30, 45]. These explain and confirm how, from a mathematical theory point of view, it is possible to find exact results about states, operators, inner products etc in a diffeomorphism invariant quantum field theory.

- There are many checks of loop quantum gravity, made by reducing the theory to special cases such as $2+1$ dimensions[46], $1+1$[47] dimensions, the linearized theory[81], with and without matter fields[48] etc. All these checks indicate that the methods and main results are reliable. The method may also be applied to Yang-Mills theory, and gives a formulation of lattice gauge theory equivalent[49] to the usual formulations. Finally, the method of loop quantum gravity can be applied to topological field theories, and reproduces results arrived at by alternative methods[46, 75].

- These methods may be applied to a wide range of theories. Many results extend to supergravity at least up till $N = 2$[50, 17, 51] and some results extend also to higher dimensional theories[52, 53]. All known kinds of matter fields may be included[48]. There are of course special results for special cases, including $3+1$ dimensions, $2+1$ dimensions etc. Thus, loop quantum gravity provides a general framework for formulating background independent quantum theories of spacetime, gravity and matter.

- This last observation leads to a strategy for finding a version of loop quantum gravity that can serve as the background independent formulation of string or $\mathcal{M}$ theory. This is based on an observation that, as in any gauge theory, the small excitations of spin foam histories around a semiclassical history will be described in terms of the embedding of a string in the spacetime[54]. There are definite proposals and results for this theory, based on a class of matrix models, closely connected to matrix Chern-Simons theory[55]. Several results exist which suggest that both perturbative string theory and a form of loop quantum gravity may be derived from such a matrix model, thus realizing the conjecture of duality between string and gauge descriptions also for quantum gravity.

- A general approach to the problems of giving a measurement theory for quantum theories of cosmology, where the observer is part of the system has been formulated in papers of Crane[56], Rovelli[57] and Markopoulou[58]. This approach, called *relational quantum cosmology* naturally incorporates a version of the holographic principle[60, 61].

It must be said that some theoretical physicists find these results surprising. It used to be argued that the perturbative non-renormalizability of general relativity implies that quantum gravity requires a modification of the principles of physics such as proposed in different ways in other approaches such as string theory, causal sets, etc. What is perhaps surprising is that loop quantum gravity has been successful, not by being radical, but by sticking rather strictly to the basic principles of general relativity and quantum theory.

Thus, the first question to be addressed is how it is that such results are possible, when the theory is nonsense when developed by traditional perturbative methods around fixed backgrounds? The answer has two parts. First the theory is completely *background independent* which means that classical spacetimes play no role in the formulation. This is necessary due to the strongly interacting nature of the Planck scale physics, the fact that the spacetime geometry is represented completely by operators, and the fact that the gauge invariances of the theory include active diffeomorphisms, which are broken by the specification of any given classical background metric[1].

The mistake all background dependent approaches make is to assume that space and spacetime have continuous, classical structures when probed at arbitrarily short wavelengths. Even rough, heuristic arguments, such as those that Wheeler and others used to give based on the uncertainty principle, suggest that this is wrong. The results of loop quantum gravity, based only on the basic principles of quantum theory and general relativity demonstrate conclusively that there is no classical spacetime manifold at Planck scales and shorter[2].

Thus, to arrive at a good quantum theory one must use methods which are background independent, and do not make the incorrect assumption that spacetime is smooth at short distances. To do this means only to take the basic principles of Einstein's theory of general relativity seriously and apply them exactly in the quantum theory. As opposed to other approaches, such as string theory, where new degrees of freedom are posited to move in fixed, classical background spacetimes, loop quantum gravity treats the geometry of spacetime at the quantum mechanical level as Einstein did at the classical level, as a completely dynamical entity. This is in fact required if the gauge invariance of the theory, which is active diffeomorphisms, is to be respected exactly in the quantum theory.

When diffeomorphism invariance is imposed exactly there is a big payoff, which is that one finds an exact description of the gauge invariant Hilbert space of quantum gravity. These are eigenstates of observables that represent the volume of spacetime and the areas of surfaces in the spacetime. These observables turn out to be finite, when regulated in a manner that respects diffeomorphism invariance. And they have discrete spectra, which demonstrates that quantum geometry is discrete at the Planck scale. These results explain, in detail, why perturbative expansions around smooth backgrounds fail; because they fail to capture any of the structure present in exact solutions of the quantum constraints that impose gauge and diffeomorphism invariance.

The second key to the success of loop quantum gravity is a property of the Einstein equations, which gives force to these general considerations of background independence and diffeomorphism invariance. This is the existence of an intimate connection between the kinematics and dynamics of general relativity and topological field theories.

A topological field theory is a field theory that has only a finite number of degrees of freedom. The few degrees of freedom it has are non-local and generally are measured either at boundaries of spacetime or by measuring phase factors or holonomies associated with loops or surfaces that cannot be shrunk to a point. Topological field theories share some properties with general relativity, such as being invariant under the gauge group of active diffeomorphisms, and being independent of any classical background[3].

At the same time, general relativity is not a topological field theory as it has an infinite number of local degrees of freedom[4] Even so, there are close connections between general

[1]For background on background independence, see [62, 63, 64].

[2]Indeed, the dynamics of Einstein's equations do not come into the derivation of the quantization of area and volume, which tell us that quantum geometry is discrete. These results are consequences only of the canonical commutation relations and the gauge invariances that define the theory. They apply whatever matter the theory is coupled to, and, with appropriate modifications, described in [17] apply also to supergravity.

[3]Some basic references on topological field theory include[65]

[4]There are other diffeomorphism invariant theories that are not topological, even some without a metric, such as Chern-Simons theory for $d \geq 5$[66] and higher form versions of Chern-Simons theory[67, 53].

relativity and topological field theory at the classical level, and these are exploited by loop quantum gravity to make a sensible quantization of general relativity. By doing so, results that at first sight are surprising and unexpected, turn out to be not only possible, but accessible with standard methods. This relationship with topological field theory in fact makes possible the key results at both the hamiltonian and path integral level that are the basis for the success of the approach.

These relationships with topological field theory are the subject of the first few sections of this paper, as they are in a good way into the subject. There are in fact three distinct ways that topological field theory enters into quantum gravity, First, the action turns out to be closely related to that of a topological field theory[3]. Second, the natural boundary term in the action, which must be added when a spacetime region with boundary is studied, is a topological field theory[13]. Third, the ground state of the theory with a non-zero Λ is derived from a topological field theory[6].

Only a few of these results are recent. The case of positive cosmological constant has, in fact, been somewhat neglected in the field in recent years. Thus it is worth mentioning some of the reasons to return to it now. These include the fact that a positive cosmological constant appears to be observed (not too mention the fact that it is in any case necessary for inflationary cosmology that the effective cosmological constant at early times be positive, and large). Beyond this, general renormalization group arguments tell us that we cannot make sense of any quantum field theory unless we include all low dimension couplings. This is certainly the case with the cosmological constant; if it is not included in a generic quantum field theory it will be there in the effective theory with a magnitude order one in either the cutoff or the supersymmetry breaking scale. This general expectation is fully born out in all the numerical work done on nonperturbative approaches to quantum gravity. This includes the early work in euclidean dynamical triangulations as well as the recent numerical work of Ambjorn, Loll and collaborators on lorentzian, or causal, dynamical triangulation models[68]. Further the latter calculations show convergence is only possible if $\Lambda > 0$. Indeed, all the evidence we have from explicit renormalization group calculations, at both the perturbative and non-perturbative levels, tells us that the theory cannot have a good low energy limit, leading to the recovery of general relativity, unless the cosmological constant term is included in the formulation of the theory.

From the point of view of loop quantum gravity, the main difference so far discovered between the theory with and without the cosmological constant is that it is only when $\Lambda \neq 0$ that the Kodama state exists. This is the only known exact solution of the theory that also is known to have a well behaved low energy description, in terms of semiclassical physics. For the case $\Lambda = 0$ the demonstration of the existence of a good low energy limit remains an open problem, although progress has been achieved recently on related technical issues such as the formulation of the renormalization group in spin foam models[69] and the construction of coherent states of the quantum gravitational field[45].

Closely connected with this issue has been a question of whether there could be long ranged correlations in the dynamics generated by the Hamiltonian constraint[70, 71]. This problem is solved for the case of $\Lambda \neq 0$, in a rather elegant fashion that will be described in section 12 [37].

As the demonstration of a good low energy limit has been a key open problem, it is of some importance that the problem is solved when $\Lambda \neq 0$. It is because of this that it can now be claimed that loop quantum gravity passes all the major tests required of a good quantum theory of gravity[5].

If further reasons after this are needed, we can mention the fact that string theory appears to have trouble with this case, so it may be one in which there is no alternative to background independent methods. Finally, this is a good case to study the problem of interpretational issues for quantum cosmology and illustrate how *relational quantum cosmology* addresses them.

This paper is written with the hope that it may serve as an introduction to this field, and for this reason the style is pedagogical and many old results are included. I assume that the reader has no prior knowledge of loop quantum gravity, and knows only the basics of Yang-Mills theory, general relativity and quantum theory. Many technical details are omitted, and may be found in the references indicated. The goal is instead to give the reader clear statements of the assumptions and results, sufficient to serve as an introduction to the field.

However, the paper is not completely a review. There are included a few new results, which support earlier claims that the theory is well defined at both the exact and semiclassical level. The new results are in sections 3, 9,10 and 13 and are also discussed in papers in preparation.

The paper has two parts, the first describes the background of classical general relativity, stressing the relationship to topological field theory. The second part describes the quantum theory of gravity with $\Lambda > 0$. The table of contents may be consulted for a more detailed description of the contents.

Part I

Classical gravity and its relation to topological field theory

2 Gravity as a gauge theory

Let us jump right in and see the power of the connection between gauge theory, gravity and topological field theory uncovered in loop quantum gravity, and then go on to show how this perspective illuminates the geometry of deSitter spacetime.

A good way into the subject is to begin with the following challenge: *Suppose that you wanted to make a theory of gravity, but you were restricted to using only the fields of an ordinary gauge theory. You are not allowed to assume the existence of a metric, either as a background or as a dynamical field. You have to work only with a gauge field. How close would you come to general relativity?*

The answer is that the simplest guess as to how to do this lands right on the nose on general relativity. Here is how this goes:

It turns out to be most direct to reason in the Hamiltonian language. From this point of view a spacetime is a manifold of the form $\Sigma \times R$, where Σ is a three dimensional manifold which will represent space, at least topologically.

[5] For a comparative evaluation of the results of the different approaches to quantum gravity, see [72].

From a hamiltonian perspective the fields we are allowed are a connection, A_a^i, and its conjugate momentum, E_i^a where a is a $3d$ spatial index and i is valued in a lie algebra, G.

Thus, we have the Poisson brackets,

$$\{A_a^i(x), E_j^b(y)\} = \delta_a^b \delta_j^i \delta^3(y,x) \tag{3}$$

We know that in a hamiltonian formulation of a gauge theory there is one constraint for each independent gauge transform[62]. The gauge invariances of a gravitational theory include at least 4 diffeomorphisms, per point. Thus,

$$I^{GR} = \int dt \int_\Sigma E^{ai}\dot{A}_{ai} - N\mathcal{H} - N^a H_a - w_i \mathcal{G}^i - h \tag{4}$$

where $\mathcal{H}_a$ generates the diffeomorphisms of Σ, $\mathcal{H}$ must be the so-called Hamiltonian constraint that generates the rest of the diffeomorphism group of the spacetime (and hence changes of the slicings of the spacetime into spatial slices) while $\mathcal{G}^i$ generates the local gauge transformations. h represents the terms in the hamiltonian that are not proportional to constraints. However, there is a special feature of gravitational theories, which is there is no way *locally* to distinguish the changes in the local fields under evolution from their changes under a diffeomorphism that changes the time coordinate. Hence h is always just a boundary term, in a theory of gravity.

From Yang-Mills theory we know that the constraint that generates local gauge transformations under (3) is just Gauss's law

$$\mathcal{G}^i = \mathcal{D}_a E^{ai} = 0 \tag{5}$$

Note that E_i^a is a vector *density*, so there is no metric used in either the Poisson brackets or Gauss's law[6].

Let us now guess the forms of the other constraints. First there must be three constraints per point that generate the diffeomorphisms of the spatial slice. Infinitesimally these will look like coordinate transformations, hence the parameter that gives the infinitesimal change is a vector field. Hence these constraints must multiply a vector field, without using a metric. Thus these constraints are the components of a one form. It should also be invariant under ordinary gauge transformations, as they commute with diffeomorphisms. We can then ask what is the simplest such beast we can make using A_a^i and E_i^a? The answer is obvious, it is

$$\mathcal{H}_a = E_i^b F_{ab}^i = 0 \tag{6}$$

where F_{ab}^i is the Yang-Mills field strength.

It is a simple exercise to show that $\mathcal{H}_a$ so defined does in fact generate a spatial diffeomorphism (plus an ordinary gauge transformation) on the fields A_a and E^a.

There remains one constraint per point, which generates changes in the time coordinate, or else in the embedding of Σ in $\mathcal{M} = \Sigma \times R$. This is called the Hamiltonian constraint. Since its action is locally indistinguishable from the effect of changing the time coordinate, it does contain the dynamics.

The Hamiltonian constraint must be gauge invariant and a scalar, since the parameter it multiplies is proportional to the local change in the time coordinate. But it could also be

[6]One thing to get used to in this field is that as there is no background metric, while in the quantum theory the metric is a composite operator, one must be completely explicit about all places the metric appears and all density weights.

a density, so we have the freedom to find the simplest expression that is a density of some weight. It turns out there are no polynomials in our fields that have density weight zero, without using a metric. But there are simple expressions that have density weight two. The two simplest such terms that can be written, which are lowest order in derivatives, are,

$$\epsilon_{abc} Tr E^a E^b E^c \quad \text{and} \quad Tr E^a E^b F_{ab} \tag{7}$$

where the Tr is in the lie algebra G. If we need to we could go to terms with more derivatives, but such terms will give trouble if we want the theory to have a simple linearization, which will be useful to reproduce Newtonian gravity and gravitational waves.

In fact these two terms already give Einstein's equations, so long as we take the simplest nontrivial choice for G, which is $SU(2)$.

Thus, we take for the Hamiltonian constraint

$$\mathcal{H} = \epsilon_{ijk} E^{ai} E^{bj} (F^k_{ab} + \frac{\Lambda}{3} \epsilon_{abc} E^{ck}) = 0 \tag{8}$$

There is a place to put a free parameter Λ which indeed will turn out to be the cosmological constant. As far as dimensions are concerned, A_a is a connection and so has dimensions of inverse length. It will turn out that E^a is related to the metric and so we should make the unconventional choice that it is dimensionless.

In fact, what we have here is Euclidean general relativity. If we want the Lorentzian theory, we need only modify what we have by putting an $\imath$ into the commutation relations, so we have instead of (3)

$$\{A^i_a(x), E^b_j(y)\} = \imath G \delta^b_a \delta^i_j \delta^3(y, x) \tag{9}$$

I have also inserted a factor of Newton's constant, G, which is necessary if E^a is dimensionless.

The Einstein's equations of course come from taking Poisson brackets with the Hamiltonian, which is a linear combination of constraints,

$$H(N, v^a) = \int_\Sigma N\mathcal{H} + v^a \mathcal{H}_a \tag{10}$$

here N and v^a are related to the usual lapse and shift, which are in turn related to the time-time and time-space components of the metric, respectively. In fact, noting that $\mathcal{H}$ has density weight two, we see that N must have density weight -1. Hence it is of the form of $g_{00}/\sqrt{detq_{ij}}$, where q_{ij} is the spatial metric. (This may seem pedantic but we will use it to good effect in the next section.)

The simplest way to evolve is with zero shift, which corresponds to the space-time components of the metric vanishing. The equations of motion are then,

$$\dot{A}_{ai} = \{A_{ai}, \int N\mathcal{H}\} = N\imath G \epsilon_{ijk} E^{bj} (2F^k_{ab} + \Lambda \epsilon_{abc} E^{ck}) \tag{11}$$

$$\dot{E}^{ai} = \{E^{ai}, \int N\mathcal{H}\} = \imath G \epsilon^{ijk} \mathcal{D}_b (N E^a_j E^b_k) \tag{12}$$

These equations, together with the seven constraints make a diffeomorphism invariant field theory whose only degrees of freedom are an $SU(2)$ connection and its conjugate electric field. To see that the theory is consistent one should check the constraint algebra, in fact it is first class. One can then count degrees of freedom and find that there are 2 canonical degrees of freedom per point. If one linearizes, one gets right away the laws for the propagation of spin 2 massless fields.

How can this be, when there is no metric in the world our equations describe? In fact there is one, it is hidden in the gauge fields. The theory we have is general relativity, with the following identifications. E_i^a is related to the three metric q_{ab} by

$$det(q)q^{ab} = E^{ai}E^{bj}\delta_{ij} \tag{13}$$

The determinant is there because the expression is a density of weight two.

The $SU(2)$ connection A_a turns out to be, for solutions, the self-dual part of the spacetime connection. For Lorentzian solutions this is complex, and its real and imaginary parts each have a geometrical interpretation.

$$A_{ai} = \text{3d spin connection}_{ai} + \frac{\imath}{\sqrt{q}} K_{ab}E_i^b \tag{14}$$

where K_{ab} is the extrinsic curvature of the 3 manifold Σ embedded in the spacetime, which in turn is essentially the time derivative of the three metric[7].

3 The deSitter solution as a gauge field

It is not of course obvious to see that the theory we have constructed is in fact Einstein's theory, or where the correspondences I've just mentioned come from. A bit later we will derive these from an action principle. But for now I want to only show how the deSitter solution fits into this framework.

We begin by noting that a family of solutions to the constraints can be read off immediately, by inspection. These are those that satisfy,

$$F_{ab}^i = -\frac{\Lambda}{3}\epsilon_{abc}E^{ci} \tag{15}$$

It is easy to see that they satisfy all seven[8] constraints. We call these *self-dual solutions* as they have the magnetic fields proportional to the electric fields.

Let us examine the simplest one of these. We can take as an ansatz that A_a^i is proportional to δ_a^i. Of course this breaks gauge invariance but this is what we have to do if we want to write an explicit solution.

As I know the answer, I will put in the right parameters:

$$A_{ai} = \imath\sqrt{\frac{\Lambda}{3}}\ f(t)\ \delta_{ai} \quad \mapsto F_{abi} = -f^2(t)\ \frac{\Lambda}{3}\epsilon_{abi} \tag{16}$$

where $f(t)$ is a function of the time coordinates.

Taking A_a^i to be purely imaginary makes sense in light of (14), it means that we are making an ansatz that the three geometry is flat, so the three dimensional spin connection vanishes. The metric can then be taken to be homogeneous, as must also be its time derivative, which is the imaginary part of A_a^i.

[7]For more details on the canonical formulation of GR in Ashtekar-Sen variables, see [4, 5] as well as the books[73].

[8]3 generate spatial diffeo's, three generate $SU(2)$ gauge transformations plus the Hamiltonian constraint.

By the self-dual initial conditions we see that

$$E^{ai} = f^2 \delta^{ai} \quad \mapsto \quad q_{ab} = f^2 \delta_{ab} \tag{17}$$

As we have satisfied the self-dual condition all the constraints are satisfied. We merely have to plug into the equations of motion (11,12) to find the evolution equations for f. Both equations of motion agree that

$$\dot{f} = N\sqrt{\frac{\Lambda}{3}} f^4 \tag{18}$$

Remembering that N is an inverse density, we should take $\mapsto N \approx det(q)^{-1/2} = f^{-3}$. This gives us

$$\dot{f} = N\sqrt{\frac{\Lambda}{3}} f^4 = \sqrt{\frac{\Lambda}{3}} f \tag{19}$$

so that $f = e^{\sqrt{\frac{\Lambda}{3}} t}$.

With the identifications we have made this gives the deSitter spacetime in spatially flat coordinates[9]:

$$ds^2 = -dt^2 + e^{2\sqrt{\frac{\Lambda}{3}}\, t}(dx^a)^2 \tag{20}$$

4 Hamilton-Jacobi theory, deSitter spacetime and Chern-Simons theory

Before we get serious and go back to the action and show why this is really Einstein's theory, there is one more simple trick we can do, which brings to light a connection between deSitter spacetime and topological field theory.

To see this connection we may begin by asking what insight Hamilton-Jacobi theory may throw on the solutions we have been considering. To use Hamilton-Jacobi theory we assume that there is a Hamilton-Jacobi functional $S(A)$ on the configuration space. As we are studying a gauge theory the configuration space is the space of the connections A_a on the three manifold Σ.

The conjugate electric field must then be the gradient of the Hamiltonian-Jacobi function,

$$E^{ai} = -\frac{\delta S(A)}{\delta A_{ai}} \tag{21}$$

We found that all seven constraints are solved with the self-dual ansatz (15). This means that the Hamilton-Jacobi function must satisfy a first order differential equation,

$$F^i_{ab} = -\frac{\Lambda}{3}\epsilon_{abc} E^{ci} = \frac{\Lambda}{3}\epsilon_{abc}\frac{\delta S(A)}{\delta A_{ci}} \tag{22}$$

This integrates immediately to

$$S_{CS} = \frac{2}{3\Lambda}\int Y_{CS} \tag{23}$$

[9] A good review of the different coordinatizations of deSitter spacetime is in [74].

Here Y_{CS} is the famous Chern-Simons invariant, given by

$$Y_{CS} = \frac{1}{2}Tr(A \wedge dA + \frac{2}{3}A^3). \tag{24}$$

It satisfies $\frac{\delta \int Y_{CS}}{\delta A_{ai}} = 2\epsilon^{abc}F^i_{bc}$

Thus, *the self-dual solutions follow trajectories in configuration space which are gradients of the Chern-Simons invariant.* Not only is deSitter spacetime one of these, there is the remarkable fact that, while there are an infinite number of self-dual solutions for Euclidean signature, there is only one for Lorentzian signature and it is deSitter spacetime.

This suggests that the semiclassical state that describes deSitter is

$$\Psi_K(A) = \mathcal{N} e^{\frac{3}{2\Lambda}\int Y_{CS}} \tag{25}$$

$\mathcal{N}$ is a normalization depending only on topology[8].

In fact, *this is an exact quantum state* as was shown in 1990 by Hideo Kodama[6]. We will return to the Kodama state and the physics that may be derived from it. But first we want to go back and find out why what we have been studying is Einstein's general theory of relativity.

5 General Relativity as a constrained topological field theory

In the last sections a very mysterious fact emerged, which is that when general relativity is written in such a way as to bring it close to gauge theory, in terms of field content and geometry, we fell upon a close relationship between an important set of solutions-the self-dual solutions, and a topological field theory. Given the ease by which topological field theories may be quantized and studied, as well as their remarkable connections with various fields of algebra, representation theory and topology, it is very important to know if this is an accident or if it has its roots in some deep relationship between gravity and topological field theory. In this section we will show that it is indeed no accident and that general relativity and topological field theory are deeply connected at the level of the action principle.

BF theory

We begin with a four dimensional topological field theory called BF theory[75]. We will work on a four manifold $\mathcal{M} = \Sigma \times R$, where Σ will be the spatial topology. There is no metric, and no other fixed background field.

We introduce now two fields. The first is an $SU(2)$ connection A^i_μ, where μ indices the spatial coordinates (to be suppressed when we use form notation and $i = 1, 2, 3$ label the generators of $SU(2)$. The second field is a two form, $B^i_{\mu\nu}$ which is also valued in the $SU(2)$ generators. To begin with we take them both real.

The action we use is,

$$I^{BF} = \int B^i \wedge F_i + \frac{\Lambda}{2} B^i \wedge B_i. \tag{26}$$

It is easy to derive the equations of motion,

$$F^i = -\Lambda B^i, \qquad \mathcal{D} \wedge B^i = 0 \tag{27}$$

We see that the curvature is constrained to be proportional to the B field, with Λ the constant of proportionality. B^i is in turn is constrained to be covariantly constant. If one counts one finds there are no local degrees of freedom, hence the theory is topological. It is also invariant under $Diff(\mathcal{M})$, the group of diffeomorphisms of the manifold. Because of the form of the action, this topological field theory is called BF theory.

Self-dualology

General relativity is in fact closely related to BF theory. To see this, we need first to understand the dynamics of general relativity in 4 spacetime dimensions in terms of self-dual and antiself-dual connections and curvatures.

Let us then have a four dimensional spacetime metric $g_{\mu\nu}$. We will at first take the spacetime to be Euclidean, then we will see how things are modified for the Lorentzian case.

It is convenient to work with frame fields, e^a_μ, with $a = 0, 1, 2, 3 = 0, i$ being four dimensional frame field indices. They are related to the metric by

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab} \tag{28}$$

with η_{ab} the flat metric on the tangent space.

Now we need do do a little *self-dualology.* Let us consider an antisymmetric tensor A_{ab} in the tangent space. Given the totally antisymmetric ϵ^{abcd} and the metric η^{ab} we may divide A_{ab} into its self-dual and antiself-dual parts

$$A^\pm_{ab} = \frac{1}{2}\left(A_{ab} \pm A^*_{ab}\right) \tag{29}$$

where $A^*_{ab} = \frac{1}{2}\epsilon_{ab}{}^{cd}A_{cd}$. We have

$$(A^\pm_{ab})^* = \pm A^\pm_{ab} \tag{30}$$

Note that these equations are consistent with $** = +1$, which is the case for Euclidean signature.

Among the objects that can be decomposed this way are the spin connection one form A_{ab} and the curvature two form $F_{ab} = dA_{ab} + \frac{1}{2}A_a{}^c A_{bc}$. These are valued in the $SO(4)$ lie algebra. The decomposition of A_{ab} into A^+_{ab} and A^-_{ab} corresponds to the Lie algebra identity $SO(4) = SO(3)_L \oplus SO(3)_R$. There are then three generators (per form index) in A^+_{ab} and they correspond to $SO(3)_L$. These three generators may then be labeled by $i = 1, 2, 3$ by the correspondence $A^+_i = A^+_{0i} = \frac{1}{2}\epsilon_i{}^{jk}A^+_{jk}$.

It is important to note that F^+_i, being also valued in $SO(3)_L$ is an $SO(3)$ gauge field which is a function only of the $SO(3)_L$ connection A^+_i.

It turns out that not only can the connection and curvature information in a four manifold be decomposed in self-dual and antiself-dual parts, the same is true for the metric information. Given the metric $g_{\mu\nu}$ one can construct three two forms from the self-dual parts of $e^a \wedge e^b$, as

$$\Sigma^i = e^0 \wedge e^i + \epsilon^i{}_{jk} e^j \wedge e^k \tag{31}$$

These forms are self-dual by construction in the internal indices. Each of the three is also *self-dual in the spacetime sense*

$${}^*\Sigma^i_{\mu\nu} \equiv \epsilon_{\mu\nu\lambda\sigma} g^{\lambda\alpha} g^{\sigma\beta} \Sigma^i_{\mu\nu} = \Sigma^i_{\alpha\beta} \tag{32}$$

From self-dual two forms to general relativity

The connection of general relativity to BF theory comes about by identifying the $SO(3)$ valued B^i fields, which are three two forms, with the self-dual two forms Σ^i corresponding to some metric $g_{\mu\nu}$. Let us see how this works.

To make the correspondence we cannot just plug

$$B^i = \Sigma^i = e^0 \wedge e^i + \epsilon^i{}_{jk} e^j \wedge e^k \tag{33}$$

into the equations of motion, (27), as the restriction (33) reduces the number of degrees of freedom per point, and there are already zero degrees of freedom per point. But we can plug (33) into the action (26) for BF theory to find that,

$$I^{JSS} = I^{BF}|_{B^i=\Sigma^i} = \int (e^0 \wedge e^i + \epsilon^i{}_{jk} e^j \wedge e^k) \wedge F_i + \frac{\Lambda}{2} \epsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \tag{34}$$

This is actually an action for general relativity[76]. In fact it is easy to see that it gives the same equations of motion as the Palatini action

$$I^{Palatini} = \int \epsilon_{abcd} \left(e^a \wedge e^b \wedge F^{cd} + \frac{\Lambda}{2} e^a \wedge e^b \wedge e^c \wedge e^d \right) \tag{35}$$

Using the projections into the self-dual and antiself-dual parts of the curvature, our strange looking action (34) can be written as,

$$I^{JSS} = \int \epsilon_{abcd} \left(e^a \wedge e^b \wedge F^{+cd} + \frac{\Lambda}{2} e^a \wedge e^b \wedge e^c \wedge e^d \right) \tag{36}$$

The equations of motion that come from varying the self-dual part of the connection, A_i^+ are

$$(\mathcal{D}^+ \Sigma)^i = 0 \tag{37}$$

These three equations are in fact the self-dual projection of the six equation of motion that corresponds to varying the Palatini action by the full $SO(3) \oplus SO(3)$ connection, A^{ab} to find,

$$\nabla e^a \wedge e^b = 0 \tag{38}$$

It is well known that the solution to this last (38) is that A^{ab} is equal to the $SO(4)$ spin connection, ω^{ab} corresponding to the frame field e^a. The solution to the equations of motion of the modified action (37) are similar, they are that A_i^+ is equal to ω^{+ab}, which is the self-dual part of the spin connection of e^a.

The other equation of motion of the Palatini equation is, with the connection taken to be the spin connection, the Einstein equations,

$$\epsilon_{abcd} \left(e^b \wedge F^{cd} + \Lambda e^a \wedge e^b \wedge e^c \wedge e^d \right) = 0. \tag{39}$$

The equation of motion of the modified equation is instead

$$\epsilon_{abcd} \left(e^b \wedge F^{+cd}(\omega^+) + \Lambda e^a \wedge e^b \wedge e^c \wedge e^d \right) = 0 \tag{40}$$

This differs from the Einstein equation (39) by a single term, which is

$$e_c \wedge F^{cd}(\omega), \tag{41}$$

but this vanishes by the Bianchi identity that sets $R_{\mu[\nu\lambda\sigma]} = 0$.

This establishes the equivalence of (36) to general relativity with Euclidean signature[10].

[10]Except that, as in the Palatini case, the fact that the action and equations of motion are polynomial

Back to BF theory

We are not yet done for as discovered, first by Plebanski[77], and then by Capovilla, Dell and Jacobson[78], we can use what we have just learned to put the action in a form that shows a direct relationship to BF theory. To do this we ask whether there are conditions on the two form fields B^i which are sufficient for there to exist a metric, and hence a frame field, e^a such that B^i are the self-dual two forms of e^a. The answer is yes, these are the five equations

$$B^{(i} \wedge B^{j)} = \frac{1}{3}\delta^{ij} B^k \wedge B_k \tag{42}$$

This is easy to see one way, by plugging in, for the other, see [78].

Why five equations? There are 18 components in the B^i's minus three gauge degrees of freedom for the $SO(3)$ rotations that mix them up, minus five equations yields the 10 components of the metric $g_{\mu\nu}$.

Thus, general relativity is the consequence of varying the BF action with the B^i fields subject to the five constraints, (42). Thus, if we add these constraints times lagrange multipliers to the BF action, we get an action for general relativity in the form,

$$I^{Plebanski} = \int B^i \wedge F_i + \frac{\Lambda}{2} B^i \wedge B_i - \frac{1}{2}\phi_{ij} B^i \wedge B^j \tag{43}$$

so long as ϕ_{ij} itself is constrained to be symmetric and tracefree.

Actually we can incorporate the cosmological constant term by requiring instead that

$$\phi_i{}^i = -\Lambda; \qquad \phi_{[ij]} = 0 \tag{44}$$

so that the action is now,

$$I^{Plebanski} = \int B^i \wedge F_i - \frac{1}{2}\phi_{ij} B^i \wedge B^j \tag{45}$$

Although we will not need it in what follows, it is interesting to note that since the action is quadratic in the B^i these can be integrated out (or solved for) to find an even simpler form for the action

$$I^{CDJ} = \int F^i \wedge F^j (\phi^{-1})_{ij} \tag{46}$$

Thus, we see that an action can be written for general relativity with a non-zero cosmological constant, *in which the metric does not appear at all.* All that appears is the curvature of the left handed part of the spin connection, and a new field ϕ_{ij}, whose trace is constrained by (44) to be the cosmological constant. The metric is instead a *composite field*, which arises only for solutions of the equations of motion.

And what is the physical interpretation of the new field ϕ_{ij}? To answer this we need only look at the field equation gotten from varying B^i in the Plebanski action, eq. (45).

$$F^i = \phi^i{}_j B^i \tag{47}$$

Since we learn by varying ϕ that there exists a metric, whose self-dual two form B^i becomes, we learn that the ϕ_{ij} are just the components of the self-dual half of the curvature two form,

means there are solutions when $det(e) = 0$ that would not be non-degenerate solutions of general relativity. Thus the space of solutions has been expanded by the addition of a kind of boundary that includes solutions with degenerate frame fields.

when expanded in components of the frame fields, or equivalently, directly in terms of the self-dual two forms of the metric. So the action (46) codes for the metric in the backhanded way that the ϕ_{ij} have to turn out to be the components of the curvatures F^i expanded in frame field components of that metric. Very sneaky, but effective, as we shall see.

The same thing with Lorentzian signature

So far everything was presented assuming the metric has Euclidean signature. But for real physics we need the metric to be lorentzian.

The same steps yield a connection between lorentzian general relativity and BF theory, but it is a bit more complicated because all the fields become complex. To understand this it is best to proceed in two steps. First, we go back and fix the definition of self-dual fields. This is necessary because, as may be easily checked, for Lorentzian signature $** = -1$. To accommodate this, we must insert an $\imath$ into the definition of self-dual tensors,

$$A^{\pm}_{ab} = \frac{1}{2}\left(A_{ab} \pm \imath A^{*}_{ab}\right) \tag{48}$$

Thus, we now have

$$(A^{\pm}_{ab})^{*} = \pm \imath A^{\pm}_{ab} \tag{49}$$

This means that the self-dual connection and curvature components A^{+}_{ab} and F^{+}_{ab} are now complex. That is, the left handed part of an $SO(3,1)$ connection is really a complex one form valued in the complexification of $SO(3)$.

Due to the $\imath$ in eq. (48) the action now has the form,

$$I^{Lorentz} = \imath \int B^i \wedge F_i - \frac{1}{2}\phi_{ij} B^i \wedge B^j \tag{50}$$

One may wonder whether the fact that A^{+} and A^{-} are both complex has doubled the degrees of freedom. The answer depends on whether or not we want the spacetime frame fields e^a to be real. If we don't require the frame fields to be real then we have extended the theory to allow all solutions of Einstein's equations where the metric is complex. In this case we have doubled the number of degrees of freedom. However, if we want the metric, and the frame field components to be real than there is a restriction on the self-dual and antiself-dual components, coming from the fact that the spin connection of the metric is real. Thus, we have

$$\bar{A}^{-}_i = A^{+}_i \tag{51}$$

This is an important difference from the Euclidean theory, in which A^{+} and A^{-} are independent, but both real.

Nevertheless, we can proceed as follows. We can consider the JSS action, (36) for the case of real e^a but complex A^{+}_i. The equations of motion then still constrain the A^{+}_i to be the self-dual parts of the spin connection and since the e^a are assumed real we still get the real lorentzian Einstein equations from (34). The only difference is there should be an $\imath$ in front of the whole action.

The next stage is to eliminate the metric completely from the action, by going to the Plebanski action (45). Here there is no simple way to incorporate the condition that the metric is real and lorentzian. The problem is that the ϕ_{ij}'s are complex for real, lorentzian metrics. The simplest thing to do seems to be to simply consider the actions (45) and (46) for complex fields A^i and ϕ_{ij}. One then gets the complexified lorentzian Einstein equations. One can then add to the field equations the initial conditions that the frame field or metric components are real. The field equations are complex, but they have the property that restricted to initial data in which the metric is real, the solutions will conserve the reality of the metric.

This may seem a strange thing to do, but from the point of view of the quantum theory it is not so bad to split the field equations into polynomial equations, which are complex, plus reality conditions on the fields. The reason is that in quantum theory the reality conditions become hermiticity conditions on operators, and these are different from the operator equations of motion, in that they involve the inner product. Strictly speaking, in quantum theory one always works with the complexified operator algebra, and imposes reality conditions through the choice of the inner product. So to do the same in quantum gravity is not very much of an innovation[11]. This is the strategy we will take up when we study the lorentzian theory.

Self-dual spacetimes and the deSitter solutions

The equations of motion gotten from the Palatini action are,

$$\mathcal{D} \wedge B = 0 \ , \qquad\qquad F^i = \phi^i_j B^j \tag{52}$$

We can see immediately the self-dual solutions.

$$\Phi^i_j = -\frac{\Lambda}{3}\delta^i_j \quad \rightarrow \quad F^i = -\frac{\Lambda}{3} B^i \tag{53}$$

This shows us how the deSitter and self-dual solutions we obtained from the Hamiltonian picture may be obtained directly as solutions to the Euler-Lagrange equations.

Derivation of the Hamiltonian formalism

Finally, we can briefly sketch how the constraints of the hamiltonian that we guessed in section 2 are derived from the forms of the action we have just described. It is easiest to work with the CDJ form of the action (46).

We begin by finding the canonical momenta, which is

$$E^{ai} = \epsilon^{abc} F^j_{bc} (\phi^{-1})^i_j \tag{54}$$

with the canonical momenta for A^i_0, E^{0i} of course vanishing.

The action can then be written as

$$I^{CDJ} = \imath \int dt \int_\Sigma E^{ai} \dot{A}_{ai} - A^i_0 \mathcal{G}_i \tag{55}$$

[11]For this reason, in the early days of loop quantum gravity the strategy of expressing the reality conditions on the metric only through the inner product, while working with a complex self-dual connection, seemed a good one as it greatly simplified the dynamics and led to many new results. More recently another alternative was adopted in many calculations, in which one worked with another $SO(3)$ connection, which is real, invented by Barbero[31]. This leads to more complicated constraint equations which, however, Thiemann showed were still manageable[30]. However, this strategy does not help in the case of the results presented here, and so is not adopted in this paper.

where the $\imath$ is there only for the Lorentzian case. Because of this the Poisson brackets for the Lorentzian case have the form eq. (9) for the Lorentzian case and eq. (3) for the Euclidean theory.

The Gauss's law constraint (5) then holds to preserve the vanishing of E^{0i}. However there are additional constraints, which arise from the fact that ϕ_{ij} is itself subject to constraints, (44), being symmetric and having trace fixed to be the cosmological constant. These must be imposed to recover the equations of motion, because without them the relationship between $\dot{A}_{ai}$ and the momenta cannot be inverted.

It is easy to check that the constraints that arise from the antisymmetric part of ϕ_{ij} vanishing is the diffeo constraint, (6), while the constraint that arises from its trace being fixed is the Hamiltonian constraint (8). Thus we arrive at the hamiltonian formulation we developed by guess work in section 2.

6 Boundaries with $\Lambda > 0$ and Chern-Simons Theory

There are several reasons we will want to consider spacetimes with boundaries. These include the important subjects of how we realize the Bekenstein bound, study the entropy of black hole and cosmological horizons and express the holographic principle. Depending on the context these boundaries will be null, as at horizons, or timelike as in the boundary of AdS spacetimes or even Euclidean, if we are working in that context. These boundaries may be at infinity, or they may have finite area.

Before we can study the quantum theory with boundaries we have to understand the role boundaries play in the classical theory. Generally when there is a boundary we will not have a sensible variational principle unless the theory is modified to take the boundary into account. Normally these modifications consist of two parts. We have to add a boundary term to the action, which just depends on the fields pulled back into the boundary. And we have to add boundary conditions. Both the boundary action and boundary conditions must be chosen so that the variations of the actions by the fields are pure bulk terms, so that the equations of motion are sensible.

Here we will consider a region of spacetime with topology $\mathcal{M} = \Sigma \times R$ with a boundary $\partial\Sigma = \mathcal{B}$. We will study only one particular class of boundary conditions, which are called *self-dual boundary conditions[13].*

The basic idea of these boundary conditions is to require that at the boundary some components of the fields satisfy the self-dual relations (15) which define the deSitter (or with $\Lambda < 0$ anti-deSitter) spacetime[13]. We cannot require all the components satisfy the self-dual conditions, otherwise only self-dual solutions will be allowed. But we can get interesting boundary conditions by requiring only a subset of the components satisfy the self-dual relations (15) when pulled back into the boundary.

We will consider cases in which the spatial components of the self-dual relation, pulled back into the boundary are satisfied, in at least one spatial slicing of the boundary. Thus, we impose,

$$F^i_{ab}|_{\mathcal{B}} = -\frac{\Lambda}{3}\epsilon_{abc}E^{ci}|_{\mathcal{B}} \tag{56}$$

There may be other components of the boundary condition, imposed on the timelike components of the boundary fields, for details about this in the euclidean case see [13], in the timelike case see [17] in the null case see [14, 15].

To complement the boundary condition we must add a boundary term to the action. The natural one to add turns out to be the Chern-Simons action of the connection A_a^i pulled back into the boundary[13]. The action then reads,

$$I^{GR} = \epsilon \int_{\mathcal{M}} B^i \wedge F_i + \phi_{ij} B^i \wedge B^j + \frac{\epsilon k}{4\pi} \int_{\partial \mathcal{M}} Y_{CS}(A) \tag{57}$$

where, from now on, $\epsilon = \imath$ for the Lorentzian theory and unity for the Euclidean theory.

There is in both cases a relation between the coupling constant, k of the Chern-Simons theory and the cosmological constant.

$$k = \frac{6\pi}{\lambda}, \qquad \lambda = \hbar G \Lambda \tag{58}$$

This ends our study of the classical physics we need to know to understand the quantum theory of gravity with $\Lambda > 0$. The key lesson of this survey is the connection to topological field theory, which we have seen arises three ways in the classical theory:

- The action for GR has the form of a constrained topological field theory.
- There is a natural class of boundary terms which require that the boundary term added to the action is the Chern-Simons action of the left handed spin connection, pulled back to the boundary.
- The deSitter and other self-dual solutions follow gradients of the Chern-Simons invariant, which can then be taken as the Hamilton-Jacobi function.

Part II
The quantum theory

7 The Kodama State

We begin with a very brief review of how diffeomorphism invariant theories are to be quantized in the Hamiltonian approach. For more details on the basic approach, see [20, 21, 73]. We do not here describe path integral methods in loop quantum gravity, but they are well developed, see, for example, [32]-[38].

7.1 A brief review of quantization

The approach taken here is Dirac quantization. This means that the whole unconstrained configuration space is quantized. This defines a *kinematical state space* $H^{kinematical}$. The constraints are imposed as operator relations on the states, as in

$$\hat{\mathcal{C}} |\Psi >= 0 \tag{59}$$

where $\hat{\mathcal{C}}$ stands for operators representing all the first class constraints of the theory. The solutions to the constraints define subspaces of the Hilbert space. A physical state must be a simultaneous solution to all the constraints.

Often this is done in two steps. The kernel of the gauge and spatial diffeomorphism constraints is called the diffeo-invariant Hilbert space, and is labeled H^{diffeo}. The simultaneous kernel of all the constraints is called the physical Hilbert space, $H^{physical}$.

Generally, new inner products need to be introduced on these Hilbert spaces, because solutions to the constraints are not normalizable in the inner products on the kinematical Hilbert space.

We will work in this and the next four chapters with the connection representation of quantum gravity[20]. After this we will switch to the loop (or spin network) representation. As in the case of the position and momentum representations in ordinary quantum mechanics these are equivalent, but complementary, in that certain calculations are easier to do in one representation than another.

Both representations are defined as representations of a certain algebra of classical observables.

From a naive point of view, we could take the canonical commutation relations (3) as the basis of the quantization. Thus, we heuristically define the connection representation by the relations

$$< A|\Psi >= \Psi(A) \quad E^{ai} = -\hbar G \frac{\delta}{\delta A_{ai}} \tag{60}$$

These satisfy the commutation relations,

$$[A^i_a(x), E^b_j(y)] = \hbar G \delta^b_a \delta^i_j \delta^3(y,x) \tag{61}$$

Note that because there is an $\imath$ in the classical commutation relation (3) no $\imath$ appears here[12]. Unless explicitly mentioned, from now on we are working with the Lorentzian theory.

However, to discuss carefully the regularization of the operators that define the theory, we need to define the quantization in terms of a different set of observables, which are the Wilson loops

$$T[\gamma, A] = TrPe^{\int_\gamma A} \tag{62}$$

in the fundamental, spin 1/2 representation, and the elements of area,

$$A[\mathcal{S}] = \int_{\mathcal{S}} \sqrt{h} \tag{63}$$

where $\mathcal{S}$ is a surface in Σ and h is the determinant of the induced metric in the surface. These have very beautiful Poisson bracket relations,

$$\{T[\gamma, A], A[\mathcal{S}]\} = \hbar G Int[\gamma, \mathcal{S}] T[\gamma, A] \tag{64}$$

where $Int[\gamma, \mathcal{S}]$ is the intersection number of the loop and the surface.

For the definitions of the connection and loop representations in terms of this algebra, see [21]. Here we will work with naive operators and mostly neglect the details of regularization procedures, which can be found in the references. However, it is very important to stress that everything said here does go through when all the details of the regularization procedures are included.

We now need the expressions of the constraints in the connection representation. These are

[12] Were we working instead with the Euclidean theory there would be an $\imath$ here.

- Gauss's law:

$$\mathcal{G}^i\Psi(A) = \mathcal{D}_a \frac{\delta}{\delta A_{ai}}\Psi(A) = 0 \tag{65}$$

- Diffeomorphism constraint

$$\mathcal{H}_a\Psi(A) = F^i_{ab}\frac{\delta}{\delta A_{bi}}\Psi(A) = 0 \tag{66}$$

- Hamiltonian constraint:

$$\mathcal{H}\Psi(A) = \epsilon_{ijk}\frac{\delta}{\delta A_{ai}}\frac{\delta}{\delta A_{bj}}(F^k_{ab} + \frac{\lambda}{3}\epsilon_{abc}\frac{\delta}{\delta A_{ck}})\Psi(A) = 0 \tag{67}$$

Note that with the ordering given here, the quantum algebra of constraints can be shown, after a suitable regularization procedure, to be consistent[5, 20, 79, 73]. This means that the commutators give terms proportional to operators, which are of the form of (new operator)$\times$ operator constraints. Thus, there are a non-trivial space of states in the simultaneous kernel of all the constraints. In fact, infinite dimensional spaces of simultaneous solutions to all the regularized constraints have been found and studied[21, 30].

For details of the different regularization procedures that can be applied to define these constraints, and the infinite dimensional families of solutions that have been found, see [21, 79, 30].

For the time being, we will be concerned with the case that Σ is compact, and without boundary. A bit later we will show how boundaries are included in the quantum theory.

7.2 The Kodama state

We will be concerned first of all with a particular simultaneous solution to all the constraints, which is the Kodama state we introduced in section 4. This is the Kodama state, defined by[6]

$$\Psi_K(A) = \mathcal{N} e^{\frac{3}{2\lambda}\int Y_{CS}} \tag{68}$$

To show that this is a solution to all the constraints, one makes use of the identity,

$$\frac{\delta\Psi_K(A)}{\delta A_{ck}} = \frac{3}{2\lambda}\epsilon^{abc}F^i_{ab}\Psi_K(A) \tag{69}$$

Thus, the Kodama state is in the kernel of the operator

$$\mathcal{J}^i_{ab} = F^k_{ab} + \frac{\lambda}{3}\epsilon_{abc}\frac{\delta}{\delta A_{ck}} \tag{70}$$

and satisfies the Hamiltonian constraint because we have chosen an ordering such that

$$\mathcal{H} = \epsilon_{ijk}\frac{\delta}{\delta A_{ai}}\frac{\delta}{\delta A_{bj}} \cdot \mathcal{J}^k_{ab} \tag{71}$$

$\mathcal{J}^i_{ab}$ is of course an operator version of the self-dual condition.

The Kodama state solves the other constraints because it is manifestly invariant under diffeomorphisms of Σ and *small* gauge transformations. (Note that only small gauge transformations are generated by constraints.)

One might think that invariance under large gauge transformations would be achieved because k is an integer. However, this would be wrong, as there is no $\imath$ in the Chern-Simons state.

Invariance under large $SU(2)$ (real) gauge transformations is instead gotten by choosing $\mathcal{N}$ to be a topological invariant also sensitive to them. For details of this, see the paper by Soo [8]. There is a reason for choosing k to be an integer, we will see it in section 11 below.

It will also be important to note that as A_a is complex, so is its Chern-Simons invariant. Hence the Kodama state is complex.

The reader may now make the following queries:

- **Does the Kodama state survive the details of a regularization procedure, needed to define the rigorous action of the constraints?**

  Yes, for details see [79].

- **Is the Kodama state normalizable?**

  Certainly it is not normalizable in the naive inner product. But this is to be expected, on two grounds. First because solutions to constraints are generally never normalizable in the inner product of the kinematical Hilbert space[73]. Second, because, as we will see below, there are components of the connection that function as a time coordinate on the configuration space[7]. The physical inner product cannot integrate over time, otherwise all energy eigenstates would be non-normalizable. Hence a new physical inner product needs to be chosen, and, given the fact that it satisfies all the physical properties we require of a physical state, it makes sense to take as a condition for the physical inner product that the Chern-Simons state, as well as its perturbations that we see below represent long wave graviton states, are normalizable.

- **But, is the Chern-Simons state really a ground state? Does it really correspond to the vacuum?**

  It is if one can study its weakly coupled excitations and they reproduce quantum field theory in curved spacetime and long wave length gravitons on deSitter spacetime.

  Note that we need only recover these for low energies in Planck units.

  We show this first for matter, then for gravitons.

8 The recovery of QFT on deSitter spacetime.

The first thing we can do to probe the Kodama state is to add matter, and then see what happens if we excite the matter in the presence of the state[13].

Adding matter fields is straightforward. In the language of loop quantum gravity it is simple to add all kinds of matter: gauge fields, fermions, scalars, antisymmetric tensor gauge fields[14]. For what we are doing here we do not need any details, so we will refer to all matter fields as ϕ, their canonical momenta as π and the matter hamiltonian as $H^{matter}(\phi, \pi)$.

[13] The material in this section comes from ref. [7] to which the reader is referred for more details.

[14] There is also no obstacle to extending the theory to supergravity, so long as $\Lambda \leq 0$ in four dimensions[50]. This has been worked out in some detail up to $N = 2$ in four dimensions. For some partial results on $d = 11$ supergravity, the interested reader can see [53].

All the constraints get new terms in the matter fields. For the Hamiltonian constraint we have

$$\mathcal{H}^{total} = \mathcal{H}^{grav}(A, E) + H^{matter}(A, E, \phi, \pi) \tag{72}$$

We will work in an extended connection representation in which the states are functionals $\Psi[A, \phi]$, π is represented by $-\imath\hbar\delta/\delta\phi$ and so forth.

As in the pure gravity case, the gauge and diffeomorphism constraints, applied to the states, require that the states are gauge invariant and invariant under diffeomorphisms of Σ.

This is straightforward, so we focus here on the hamiltonian constraint.

To study perturbations of the Kodama state,we follow the proposal of Banks[80], which is to study the semiclassical approximation in quantum cosmology by a version of the Born-Oppenheim approximation, in which the gravitational degrees of freedom play the role of the heavy, nuclear degrees of freedom, while the matter degrees of freedom play the role of the light, electron degrees of freedom.

Thus, we consider a product state of the form

$$\Psi(A, \phi) = \Psi_K(A)\chi(A, \phi) \tag{73}$$

The exact Hamiltonian constraint is then of the form

$$\left(H^{grav} + H^{matter}\right)\Psi_K(A)\chi(A, \phi) = 0 \tag{74}$$

The idea is to make an approximation to the exact equations, which is described in terms of quantum matter fields propagating on a classical background spacetime (A^0, E^0). This approximation is gotten by expanding the Wheeler-DeWitt equation (74) in a neighborhood of a classical solution. We use the fact that the Kodama state can be understood as a WKB state as well as an exact solution. This tells us that the classical background (A^0, E^0) must be deSitter spacetime, as it is the unique solution gotten by taking S_{CS} to be the Hamiltonian-Jacobi function, consistent with the requirement that the lorentzian metric be real.

We will describe the details of this approximation, for the case of a scalar field, in section 10. As a prelude, we mention here the basic features of the results.

As shown in [7], we find that an approximation to (74) takes the form of a Tomonaga-Schwinger equation:

$$\imath\frac{\delta\chi}{\delta\tau_{CS}} = \frac{1}{\Lambda}H^{matter}_{E^{ai}=(3/\Lambda)\epsilon^{abc}F^i_{bc}}\chi + O(l_{Pl}E) \tag{75}$$

In this equation, the matter Hamiltonian is evaluated with classical gravitational fields satisfying the self-dual condition $E^{ai} = (3/\Lambda)\epsilon^{abc}F^i_{bc}$. As we just said, the reality conditions then tell us that the background is deSitter. We have neglected higher order terms in $l_{Pl}E$, where E is the energy of the matter fields measured with respect to the background metric.

The approximation procedure picks out a time coordinate called τ_{CS}, related to the Chern-Simons invariant. It is first of all a coordinate on the configuration space of the theory, defined by

$$\delta\tau_{CS}(x) = \frac{1}{2}\mathcal{I}m\ \epsilon^{abc}F^i_{bc}\delta A_{ai}(x) \tag{76}$$

Thus, integrated over the spatial manifold Σ, we have

$$\int_\Sigma \delta\tau_{CS}(x) = \delta\mathcal{I}m\int Y_{CS}(A) \tag{77}$$

If we take the integral, we can define

$$T_{CS}=\int_\Sigma \tau_{CS}=Im\int_\Sigma Y_{CS} \tag{78}$$

This can be argued to provide a provides a good global time parameter on the configuration space[7, 90]. This is because its derivative is always orthogonal, in the tangent space of the configuration space, to both the gauge directions and the directions that parameterize the physical degrees of freedom.

When evaluated on a background solution, this gives rise to a time coordinate on the spacetime. One can then show that, to leading order in λ, $Y_{CS}=\imath\sqrt{det(q)}\mathcal{K}+O(\sqrt{\lambda})$, where $\mathcal{K}$ is the trace of the extrinsic curvature K_{ab}. Thus. this choice of time coordinate agrees, to leading order in λ, with that proposed by York[7]. This time coordinate has been shown to have many good properties that an intrinsic time coordinate should have.

Thus, QFT on deSitter *is* a good approximation to the physics of $\Psi(A,\phi)=\Psi_K(A)\chi(A,\phi)$ when $\lambda=\hbar G\Lambda$ and $l_{Planck}E$ are small. This stands as a first piece of evidence that $\Psi_K(A)$ may be indeed a good ground state.

There are additional terms in $l_{Planck}E$, where E is the matter energy on the deSitter background. We will study the effect of these terms in section 10.

9 Gravitons from perturbations around the Kodama state

To further probe the properties of the Kodama state we should also investigate its gravitational excitations. To do this we return to the case of pure gravity and consider states of the form

$$\Psi[A]=\mathcal{N}e^{\frac{3}{2\lambda}\int Y_{CS}+\lambda S'(A)} \tag{79}$$

It is not difficult to show that there are solutions of this form, and that they do describe long wavelength gravitons moving on the classical background of deSitter spacetime[15]. But to do this we first need to know how to recognize gravitons in this language. We then detour to summarize the results in this area,

Linearized gravity on a deSitter background

The quantization of linearized general relativity in Sen-Ashtekar variables of the kind we are using was considered early in the study of loop quantum gravity[81]. There a complete description was obtained of gravitons on a Minkowski spacetime background. It is trivial to extend what was done there to gravitons moving on a deSitter background. As we want results that hold for small λ, it is convenient to use λ as an expansion parameter.

Thus, we expand classical general relativity around the deSitter background studied in section 3

$$A_{ai}=\imath\sqrt{\Lambda}f(t)\delta_{ai}+\lambda a_{ai};\quad E^{ai}=f^2\delta^{ai}+\lambda e^{ai} \tag{80}$$

It is trivial then to compute the constraints to linear order to find the linearized constraints satisfied by the a_{ai}'s and e^{ai}'s. To solve them we need to impose 7 gauge fixing

[15]More details concerning these results will be reported elsewhere[103, 82].

conditions. A natural set to impose is[16]

$$a_{[ai]} = a^a_a = \partial_a a^a_i = 0 \tag{81}$$

where indices are raised and lowered by the background metric, q^0_{ab}. The simultaneous solution of the linearized constraints and gauge fixing conditions is

$$\partial_a e^a_i = e_{[ai]} = e^a_a = 0. \tag{82}$$

The result is that the theory is reduced to a^r's and e^r's that are tracefree, divergence free and symmetric. These are spin two fields.

The linearized poisson brackets can be derived by applying the full Poisson brackets to the linearized fields. This gives,

$$\{a^r_{ai}(x), e^{bj}_r(y)\} = \imath P^{bj}_{ai}\delta^3(x,y) \tag{83}$$

where P^{bj}_{ai} is the projection operator onto the symmetric, transverse, tracefree fields.

Finally, we have to construct the linearized hamiltonian. This comes from the quadratic terms in the integral of the hamiltonian constraint, and comes out to be,

$$h(a^r, e^r) = f^{-1}\left[\epsilon^{ajk}(\mathcal{D}^0_a a^r_{bk})e^b_{r\ j} + \Lambda e^{ai}_r e_{r\ ia}\right] \tag{84}$$

It is then straightforward to quantize this theory, yielding a quantum theory of gravitons on the background of deSitter spacetime.

Linearization of the exact quantum theory agrees with the quantization of the linearized theory, for long wavelength.

Now we want to go the other way around and study expansions of exact states in powers of λ around the Kodama state. We consider a product state of the form (79) and solve all seven constraints in a neighborhood of the classical trajectory on the configuration space.

The 6 kinematical constraints give:

$$\int_\Sigma (\mathcal{D}_a w)^i \frac{\delta S'}{\delta A_{ai}} = 0; \quad \int_\Sigma v^a F^i_{ab}\frac{\delta S'}{\delta A_{ai}} = 0 \tag{85}$$

where w^i and v^a are arbitrary functions on Σ. These are linearized around the dS background. They are solved by taking $S' = S'(f, a_{ai})$ with a_{ai} symmetric and transverse wrt the dS background. Thus,

$$\frac{\delta S'}{\delta A_{ai}} = \frac{\imath}{\sqrt{\Lambda}}\delta_{ai}\frac{\delta S'}{\delta f} + \frac{1}{\lambda}\frac{\delta S'}{\delta a_{ai}} \tag{86}$$

The hamiltonian constraint is,

$$\epsilon_{abc}\epsilon_{ijk}\frac{\delta}{\delta A_{ai}}\frac{\delta}{\delta A_{bj}}\left[\frac{\delta S'}{\delta A_{ck}} e^{\frac{3}{2\Lambda}\int Y_{CS} + S'(A)}\right] = 0 \tag{87}$$

Using (86), this can be expanded to give:

$$\imath\frac{\partial S'}{\partial t} = \hat{H}^2 S' + O(l_{Planck}E) + O(\sqrt{\lambda}) \tag{88}$$

[16] As an exercise one has to check that these seven gauge fixing conditions do together with the linearized constraints make a second class algebra.

where the free Hamiltonian is

$$\hat{H}^2 = \hat{h}(\hat{a}, \hat{e} = \frac{\delta}{\delta a}) \tag{89}$$

and E is the energy of the graviton state with respect to the background. Thus we conclude that for long wavelength perturbations, but only so long as $l_{Pl}E << 1$, the linearized theory is recovered. However it must be emphasized that we have only obtained a correspondence with the standard linearized theory for low energy and small λ.

To go beyond this, which will be necessary if we want to compute graviton graviton scattering, we need to have exact expressions for the physical states which are obtained by local perturbations of the Kodama state. This requires solving the kinematical constraints exactly, rather than order by order in l_{Pl} and λ. To do this we must go to the loop representation. This will be the subject of sections 12 and 13. However, before going there there are a few more things of physical interest we can learn from the connection representation.

10 Corrections to energy momentum relations

Now that we have recovered known physics from the Kodama solution, we may go on to see if the theory makes any predictions beyond the recovery of quantum field theory in the semiclassical limit. To see that it may, let us look in detail at the fundamental equation (74). For simplicity we consider the case of a massless, non-interacting scalar field, although similar conclusions apply for other matter fields[17]. For this case the form of the matter term in the Hamiltonian constraint is

$$H^{matter}(x) = \frac{G\hbar}{2}\left(\pi^2 + (\partial_a\phi)(\partial_b\phi)E^{ai}E^b_i\right) \tag{90}$$

where π is the canonical momentum of the scalar field. Implemented as a quantum operator this is,

$$\hat{H}^{matter}(x)\Psi_K(A)\chi(A,\phi) = \frac{\hbar G}{2}\left(\pi^2 + (\hbar G)^2(\partial_a\phi)(\partial_b\phi)\frac{\delta}{\delta A_{ai}}\frac{\delta}{\delta A^i_b}\right)\Psi_K(A)\chi(A,\phi) \tag{91}$$

In section 8 we recovered quantum field theory in curved spacetime from an approximation to this last expression. In this approximation we considered only the terms in which the factors of $\hat{E}^{ai} = -\hbar G\delta/\delta A_{ai}$ in the second term act on the Kodama state, giving terms proportional to the background frame field , $\delta_{ai}f^2$. Keeping only these terms we have

$$\hat{H}^{matter}(x)\Psi_K(A)\chi(A,\phi) = \frac{1}{2}\left(\pi^2 + \delta^{ab}f^4(\partial_a\phi)(\partial_b\phi)\right)\chi(A,\phi) \tag{92}$$

which is the hamiltonian for the scalar field on the background spacetime.

To go beyond the semiclassical approximation we may then consider the other terms in (91) in which one or both of the functional derivatives by A_{ai} act directly on the perturbed state $\chi(A,\phi)$. These still give terms linear in χ so they may be interpreted as corrections to the functional Schroedinger equation. We will see that these terms give predictions of new physics.

[17] More details concerning the results of this section will appear in [82].

The interpretation of the new terms is easiest when we can neglect the effect of the cosmological constant, and approximate a region of deSitter spacetime by a region of flat spacetime. To get predictions for the theory in flat spacetime, we proceed in two steps. First we evaluate the approximate solutions to the Wheeler deWitt equation at the background values of the connection and metric we studied in section 3. These are given by eqs. (16,17). We then approach flat spacetime by neglecting terms such as $k^2\Lambda$, where k is the momentum of a particle, which vanish in the limit $\Lambda \to 0$. This is of course a good approximation in the observed situation in which the cosmological constant is non-zero, but very small.

Evaluating the action of $\delta/\delta A_{ai}$ to leading order on $\Psi_K(A)$, we found that,

$$\hat{E}^{ai}(x)\Psi_K(A) = -\hbar G \frac{\delta \Psi_K(A)}{\delta A_{ai}(x)} = f^2(t)\delta_{ai}\Psi_K(A) \tag{93}$$

Thus, the full action of the functional derivatives gives

$$\begin{aligned} \frac{(\hbar G)^2}{2}(\partial_a\phi)(\partial_b\phi)\left(\frac{\delta}{\delta A_{ai}}\frac{\delta}{\delta A^i_b}\right)\Psi_K(A)\chi(A,\phi) &= \Psi_K(A)\left\{\frac{f^4}{2}(\partial_a\phi)(\partial_b\phi)\delta^{ab}\chi(A,\phi)\right. \\ &+ \left.(\partial_a\phi)(\partial_b\phi)[-\hbar G f^2\frac{\delta\chi(A,\phi)}{\delta A_{ab}} + \frac{(\hbar G)^2}{2}\frac{\delta^2\chi(A,\phi)}{\delta A_{ai}\delta A^i_b}]\right\} \end{aligned} \tag{94}$$

In section 8 we also saw that the time derivative in the functional Schroedinger equation came from terms in which a single derivative in $\frac{\delta}{\delta A_{ai}}$ in the gravity part of the hamiltonian constraint acted on $\chi(A,\phi)$, while the remaining functional derivatives act on $\Psi_K(A)$ giving factors of the background fields through (93). But there are also terms in the gravity part in which two and three functional derivatives act on $\chi(A,\phi)$. This will give additional corrections to the functional Schroedinger equation.

Before writing them all out we have to consider the effect of the gauge and diffeomorphism constraints. By an analysis similar to the one of the last section, they tell us that

$$\chi(A,\phi) = \chi(\tau, a_{ai}, \phi) \tag{95}$$

where as before a_{ai} is transverse and tracefree. The dependence on the a_{ai} describes the couplings to gravitons.

τ is a field that parameterizes the trace part of the background A_{ai} and is given by

$$A_{ai}(x) = \delta_{ai} e^{\sqrt{\frac{\Lambda}{3}}\tau(x)} + \ldots \tag{96}$$

For the background A_{ai} discussed in section 3 we have $\tau(x) = t$. However its important to keep the distinction clear: while t is a coordinate on a particular classical solution, $\tau(x)$ is a field that parameterizes the trace part of A_{ai} on the whole configuration space.

On the solution, τ is related to the Chern-Simon time described in section 8 by

$$\tau_{CS}(x) = \left(\frac{\Lambda}{3}\right)^{3/2} e^{3\sqrt{\frac{\Lambda}{3}}\tau(x)} \tag{97}$$

Thus, we have

$$\hat{E}^{ai}\chi(A,\phi) = -\hbar G\frac{\delta\chi(A,\phi)}{\delta A_{ai}} = \frac{\imath\hbar G}{\Lambda}\delta_{ai}\frac{\delta\chi(A,\phi)}{\delta\tau} - \hbar G\frac{\delta\chi(A,\phi)}{\delta a_{ai}} \tag{98}$$

We are interested in finding leading order corrections to the propagation of a free field on the background spacetime. Thus, we can neglect the couplings to gravitons. Doing so gives us corrections to the Tomonaga-Schwinger equation:

$$\begin{aligned}
\imath f^4 \frac{\delta\chi(\tau,\phi)}{\delta\tau(x)} &= \frac{1}{2}[\pi^2 + f^4(\partial_a\phi)^2]\chi(\tau,\phi) \\
&+\frac{1}{2}(\partial_a\phi)^2 \left[\frac{2\imath\hbar G f^2}{\Lambda}\frac{\delta}{\delta\tau(x)} - \left(\frac{\hbar G}{\Lambda}\right)^2 \frac{\delta^2}{\delta\tau^2(x)}\right]\chi(\tau,\phi) \\
&+\left[2\frac{\hbar G f^2}{\Lambda}\frac{\delta^2}{\delta\tau^2(x)} + \imath\left(\frac{\hbar G}{\Lambda}\right)^2 \frac{\delta^3}{\delta\tau^3(x)}\right]\chi(\tau,\phi) \qquad (99)
\end{aligned}$$

The corrections on the second line come from the action of $\frac{\delta}{\delta A_{ai}}$ on χ from the matter hamiltonian density, while the corrections on the last line come from the higher order terms in $\frac{\delta}{\delta A_{ai}}$ from the gravitational part of the hamiltonian constraint.

To see what the effect of the corrections is on ordinary physics, we have to re-express the Tomonoaga-Schwinger equation in terms of measurable quantities that govern the low energy physics. One way to approach this is the following.

We are interested in extracting quantum field theory on Minkowski spacetime, in the limit $\Lambda \to 0$. For the limit to be non-singular we must rescale the time coordinate, because of the factors of $\hbar G/\Lambda$ in front of the $\delta/\delta\tau$ derivatives. In any case we need to rescale to remove a density factor, as we are interested in expressing the final answer in terms of a Schrodinger equation rather than a Tomonaga-Schwinger type equation. To do this we must replace the functional degree of freedom $\tau(x)$, which we have chosen to represent time by a global coordinate T. This coordinate T is taken to be proportional to τ on a τ = constant slice. However $\delta/\delta\tau(x)$ and $\partial/\partial T$ have different density weights and dimensions and this must be compensated for.

We accomplish both if we rescale so that on a fixed τ = constant slice,

$$\frac{\hbar G}{\Lambda}\frac{\delta}{\delta\tau(x)} = \alpha l_{Pl}\sqrt{detq^0_{ab}}\frac{\partial}{\partial T} \qquad (100)$$

where α is a dimensionless parameter. The factor of $\sqrt{detq^0_{ab}}$ is due to the fact that $\delta/\delta\tau(x)$ is a density. This form is required as l_{Pl} is the only dimensional parameter in the theory when $\Lambda \to 0$. We will see shortly how α is fixed.

The next step is to integrate over the spatial manifold, so as to recover the Schroedinger equation. To do this we multiply the whole expression by $1/\sqrt{detq^0_{ab}}$, because the form of the hamiltonian constraint we are using has density weight two, and then integrate. We set $f = 1$ as we are about to take $\Lambda \to 0$ and we note that in the coordinates we are using $det(q^0_{qb}) = 1$. This gives us,

$$\begin{aligned}
\imath\frac{\partial\chi(T,\phi)}{\partial T}\left(\frac{\alpha V\Lambda}{l_{Pl}}\right) &= \int_\Sigma \frac{1}{2}[\pi^2 + (\partial_a\phi)^2]\chi(T,\phi) \\
&+\int_\Sigma \frac{1}{2}(\partial_a\phi)^2\left[2\imath\alpha l_{Pl}\frac{\partial}{\partial T} - \alpha^2 l_{Pl}^2\frac{\partial^2}{\partial T^2}\right]\chi(T,\phi) \\
&+\left(\frac{\alpha V\Lambda}{l_{Pl}}\right)\left[2\alpha^2 l_{Pl}^2\frac{\partial}{\partial T} + \imath\alpha^3 l_{Pl}^3\frac{\partial^2}{\partial T^2}\right]\chi(T,\phi) \qquad (101)
\end{aligned}$$

where V is the volume of the spatial manifold according to the background metric, $V = \int_\Sigma \sqrt{det(q^0)}$. We impose an infrared cutoff so V is finite. We will shortly take $V \to \infty$ as $\Lambda \to 0$.

However before we do this we should take into account the renormalization between the bare fields and the physical fields that enter into the low energy physics. We expect to have to renormalize because there are interactions between the scalar and gravitational fields. However, as there is a cutoff on the spatial resolution in the exact diffeomorphism invariant states[18] we expect the wavefunction renormalization to be finite and to be proportional to powers of the ratio $\frac{L}{l_{Pl}}$, with $V = L^3$ as they represent infrared and ultraviolet cutoffs. Further as we expect relativistic invariance to hold at least up to corrections in l_{Pl}, we expect that $\pi \approx \dot\phi$ and $\partial_a\phi$ to renormalize by the same factor, again up to possible corrections in l_{Pl}[19]. Thus we expect

$$\pi = Z\pi_R, \qquad \partial_a\phi = Z\partial_a\phi_R \tag{102}$$

where Z is a multiplicative renormalization. Let us suppose that $Z = \beta^{1/2}(L/l_{Pl})^{d/2}$. As the background represents deSitter spacetime, it is natural to scale $\Lambda = \gamma/L^2$ where γ is a factor of order one depending on the topology. Thus we have

$$\begin{aligned} \imath\frac{\partial\chi(T,\phi)}{\partial T}\frac{\alpha\gamma}{\beta}\left(\frac{l_{Pl}}{L}\right)^{(d-1)} &= \int_\Sigma \frac{1}{2}[\pi_R^2 + (\partial_a\phi_R)^2]\chi(T,\phi) \\ &+ \int_\Sigma \frac{1}{2}(\partial_a\phi_R)^2\left[2\imath\alpha l_{Pl}\frac{\partial}{\partial T} - \alpha^2 l_{Pl}^2\frac{\partial^2}{\partial T^2}\right]\chi(T,\phi) \\ &+ \frac{\alpha\gamma}{\beta}\left(\frac{l_{Pl}}{L}\right)^{(d-1)}\left[2\alpha^2 l_{Pl}^2\frac{\partial^2}{\partial T^2} + \imath\alpha^3 l_{Pl}^3\frac{\partial^3}{\partial T^3}\right]\chi(T,\phi) \end{aligned} \tag{103}$$

We must recover the Schroedinger equation in the limit $L \to \infty$, $l_{Pl} \to 0$. As the renormalized Hamiltonian

$$H_R = \int_\Sigma \frac{1}{2}[\pi_R^2 + (\partial_a\phi_R)^2] \tag{104}$$

should generate evolution in T, we require that the coefficient of $\imath\frac{\partial}{\partial T}$ on the left hand side be unity. This tells us that

$$\alpha = \frac{\beta}{\gamma}\left(\frac{L}{l_{Pl}}\right)^{(d-1)} \tag{105}$$

The limit then exists for $d \leq 1$. If $d < 1$ the additional terms disappear, and the usual Lorentz invariant quantum field theory is recovered. But in the case that $d = 1$ we have,

$$\alpha = \frac{\beta}{\gamma} \tag{106}$$

is a factor of order unity. Then our equation is

$$\begin{aligned} \imath\frac{\partial\chi(T,\phi)}{\partial T} &= H_R\chi(T,\phi) \\ &+ \int_\Sigma \frac{1}{2}(\partial_a\phi_R)^2\left[2\imath\alpha l_{Pl}\frac{\partial}{\partial T} - \alpha^2 l_{Pl}^2\frac{\partial^2}{\partial T^2}\right]\chi(T,\phi) \\ &+ \left[2\alpha^2 l_{Pl}^2\frac{\partial^2}{\partial T^2} + \imath\alpha^3 l_{Pl}^3\frac{\partial^3}{\partial T^3}\right]\chi(T,\phi) \end{aligned} \tag{107}$$

[18] See the sections on the loop representation.

[19] In the following we ignore such corrections, but if found by calculations they can be inserted directly in the following expressions.

Thus, under the assumptions stated, we predict corrections to the Schroedinger equation, of order l_{Pl}, with the finite dimensionless coefficient α determined by the wavefunction renormalization of the scalar field theory interacting with gravity.

Now, to analyze the scalar field theory we can use to a first approximation a regular Fock space quantization in which

$$\phi(x,t) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega}} \left[a_k f_k(x) e^{-\imath\omega t} + a_k^\dagger f_k^*(x) e^{\imath\omega t} \right] \tag{108}$$

where, as $\Lambda \to 0$, $f_k(x) \approx e^{\imath k_a x^a}$.

The Fock space one particle states are not going to be exact solutions to the Wheeler-DeWitt equation, but we can search for solutions of the form

$$\chi(T,\phi) = e^{-\imath\omega T} |k> + O(l_{Pl}) \tag{109}$$

where $|k>$ is a one particle Fock state. We do not set $\omega = |k|$ in (108), instead we take the components $< k| \dots |k >$ of the Wheeler-DeWitt equation to extract the relation between ω and $|k|$.

We find, after the standard normal ordering

$$< k| : H_R : |k >= \frac{\omega^2 + k^2}{2\omega} \tag{110}$$

$$< k| : \int_\Sigma (\partial_a \phi_R)^2 : |k >= \frac{k^2}{2\omega} \tag{111}$$

Applying eq. (107) to this state we find

$$\omega^2 \frac{(1 + 4\alpha l_{Pl}\omega + 2\alpha^2 l_{Pl}^2 \omega^2)}{\left(1 + \alpha l_{Pl}\omega + \frac{1}{2}\alpha^2 l_{Pl}^2 \omega^2\right)} = k^2 \tag{112}$$

We thus see that there are corrections to the energy momentum relations. These kinds of corrections have been discussed by a number of authors, and with α of order unity are expected to be measurable in experiments involving gamma ray busts and cosmic rays. (For details see [9, 10, 11, 12].) For example we predict that a massless scalar particle travels with a speed

$$v = \frac{d\omega}{d|k|} = 1 - \frac{3}{2}\alpha l_{Pl}\omega + \dots \tag{113}$$

If the same effect occurs for photons it is expected in the near future that it will be detectable in timing experiments in gamma ray bursts[9, 12]. It is straightforward to extend these results to photons and other fields, these results will be reported elsewhere.

It is very interesting to note that similar corrections have been derived by Gambini and Pullin[83] and by Alfaro, Morales-Tecotl and Urrutia[84] from loop quantum gravity, using a different approach in which they study the propagation of matter fields on background spatial geometries represented by “weave states.” They concluded that the exact value of the coefficient α depends on the details of the wavefunction of the ground state, here, having chosen a particular groundstate we find a prediction for α involving also the wavefunction renormalization of the matter field.

These corrections raise several intriguing issues, which are studied in some detail in [82]. Among the issues they raise is that of Lorentz invariance. Of course one possibility is that

Lorentz invariance is simply broken. A possible source of the breaking of Lorentz invariance in this case is that we studied the Wheeler DeWitt equation in the neighborhood of the connection described in section 3. This gives the spatially flat coordinatization of a region of deSitter spacetime, where the region coordinatized is the interior of a horizon of a single inertial observer.

However, we might have considered a semiclassical approximation to the Wheeler-deWitt equation corresponding to a global coordinatization of deSitter spacetime. Alternatively, we might have considered expanding around a semiclassical approximation to a coordinatization of the region accessible to any inertial observer in deSitter spacetime. Thus, we might have expected relativistic invariance to be restored in the limit $\Lambda \to 0$.

To study this issue we must go beyond the approximation of quantum field theory on fixed backgrounds, as the quantum fields defined in different causal regions of a fixed spacetime background are generally not unitarily equivalent. This raises a host of issues, but they should be resolved by the full quantum theory of gravity. As we have in fact been studying a well defined approximation to the full theory, this should in the future be possible.

We may note that one possible outcome of such a study is that the principle of the relativity of inertial observers is maintained, but the lorentz transformations become non-linear when energies and momenta are of the order of the Planck scale. This possibility has been studied by a number of authors[85, 86, 87, 88]. We may note that the deformed energy momentum relation we arrived at here is of a form that would be permitted in such a modified formulation of the Lorentz transformations[88].

11 The thermal nature of quantum gravity with $\Lambda > 0$.

It is well established that quantum field theory on deSitter spacetime must be interpreted as irreducibly thermal[89], with a temperature given by

$$\mathcal{T} = \frac{1}{2\pi}\sqrt{\frac{\Lambda}{3}} \tag{114}$$

This can be understood as due to the presence of the horizon. Alternatively, one can show that any quantum field on deSitter spacetime, satisfies the KMS condition for a thermal state. This is that the continuation of any correlation function to imaginary time coordinate $t_E = \imath t$ be periodic, with period $\beta = 1/\mathcal{T}$.

What is not so well known, however, is that in loop quantum gravity the full background independent quantum theory of gravity plus arbitrary matter fields has also an irreducibly thermal nature. This is because it satisfies the KMS condition on the whole configuration space[20].

To apply the KMS condition to quantum gravity we need two things: 1) a definition of a time coordinate on the configuration space and 2) a definition of the continuation to Euclidean time. In a background independent theory we cannot use a time coordinate on a given classical spacetime, as that is just a given classical solution. We need instead a time coordinate on the configuration space of the theory.

We saw above in section 8 that there is in fact a preferred time coordinate on the configuration space, which is picked out by the semiclassical expansion around the Kodama state.

[20]The argument of this section is taken from [7].

It is equal to the imaginary part of the Chern-Simons invariant, eq. (78). There are other arguments that confirm that (78) is a good time coordinate on the configuration space, for example it is always normal to the gauge directions in the tangent space of the configuration space. For details see [7, 90]. The Chern-Simons time coordinate is dimensionless, as we saw in the last sections when we evaluate it on a given solution we have to scale it appropriately.

It is interesting to note that (Lorentzian) Kodama state can be written

$$\Psi_K(A) = e^{\imath M T_{CS}} e^{\frac{k}{4\pi}\mathcal{R}e \int Y_{CS}(A)} \tag{115}$$

where the dimensionless "energy" is

$$M = \frac{k}{4\pi} = \frac{3}{2\lambda} \tag{116}$$

Now we need a definition of how to continue to Euclidean time. As we are dealing with a theory of spacetime we should continue the whole theory to Euclidean signature. This requires the following changes: The connection A_{ai} becomes a real, $SU(2)$ connection and there is now an $\imath$ in $E^{ai} = \imath\delta/\delta A_{ai}$. As a consequence of which the Chern-Simons state is now,

$$\Psi_K^{Euc}(A) = e^{\frac{\imath k}{4\pi}\int Y_{CS}(A)} \tag{117}$$

The Euclidean time coordinate is then just

$$T_{ECS} = \int Y_{CS}(A) \tag{118}$$

as can be seen directly, or by repeating the derivation from the semiclassical theory. Thus, the Euclidean wavefunction is,

$$\Psi_K^{Euc}(A) = e^{\imath \frac{k}{4\pi} T_{Euc}} \tag{119}$$

This is periodic in T_{Euc}. However, this is not enough to show that the KMS condition is satisfied, for that requires that *every correlation function* be periodic.

Interestingly enough, this is in fact the case whenever Σ is chosen so that $\pi^3(\Sigma)$ is nontrivial. In this case there are large gauge transformations that have the property that

$$\int Y_{CS}(A) \to \int Y_{CS}(A) + 8\pi^2 n \tag{120}$$

where n is the winding number of the large gauge transformation. This means that $T_{ECS} = \int Y_{CS}(A)$ is actually a periodic function on the configuration space. As a result, *every* correlation function will satisfy the KMS condition in T_{ECS}, no matter what the state. That is, by equating configurations of A_{ai} that differ by a large gauge transformations we reduce the topology of the configuration space to a circle, which is parameterized by T_{ECS}.

As a result of this universal periodicity there is a temperature, given in dimensionless units by $\mathcal{T}_{dimless} = \frac{1}{8\pi^2}$. This dimensionless temperature corresponds to the fact that the time coordinate on the configuration space, T_{CS} is dimensionless.

It is intereseting to ask if this dimensionless temperature corresponds to the temperature on deSitter spacetime. To investigate this we may consider a trajectory in configuration space that corresponds to a slicing of deSitter spacetime with topology $S^3 \times R$. Such coordinates are given by

$$ds^2 = -(1 - \frac{\Lambda r^2}{3})dt^2 + \frac{1}{(1 - \frac{\Lambda r^2}{3})}dr^2 + d\Omega^2 \tag{121}$$

To work out the scaling of the coordinate t on the solution with the coordinate T_{CS} on the configuration space, we compute

$$\frac{\partial T_{CS}}{\partial t} = \int_{S^3} N\{T_{CS}(A), \mathcal{H}\} \tag{122}$$

where the (densitized) lapse N is read off from the solution (121). A simple calculation gives

$$\frac{\partial T_{CS}}{\partial t} = 4\pi\sqrt{\frac{\Lambda}{3}} \tag{123}$$

Thus, if the Euclidean continuation T_{ECS} is periodic with period $8\pi^2$, the Euclidean continuation of the time coordinate on the solution must be periodic with period $2\pi\sqrt{\frac{3}{\Lambda}}$. In fact, this is the periodicity of the Euclidean deSitter solution, in these coordinates! This leads to the temperature of deSitter spacetime, (114).

Thus, we learn that *the periodicity of the Euclidean deSitter spacetime is a consequence of that spacetime having an interpretation as a trajectory on the configuration space of* $SU(2)$ *connections.* The periodicity of the Euclidean Schwarzschild solution is a consequence of the fact that the whole configuration space is periodic due to the action of the large gauge transformations. This is yet another connection between the properties of the gauge theory and the physics of gravitation. Thus, the thermal nature of quantum field theory on deSitter spacetime is a consequence of a deeper and more general result, which is that the whole quantum theory with $\Lambda > 0$ is thermal.

Finally, we can deduce one more fact from these considerations. For the analysis we have just given to be relevant to the Kodama state, it must be that the Euclidean Kodama state is itself well defined under large gauge transformations. This will only be the case if k is an integer.

12 The loop transform for positive Λ

To go beyond a semiclassical expansion we need to study the exact quantum states of the theory. This can be done using the loop, or as it is sometimes called, the spin network, representation of the state space. Here we discuss the basics of the loop representation, paying special attention to the modifications required for $\Lambda \neq 0$.

The loop representation can be defined in different ways, all leading to the same results. The most secure way is to construct it directly as a representation of the loop algebra, equation (64). Alternative constructions using rigorous methods involving measures on the space of connections are described in [45]. Here we will use the original technique, which is to construct the loop representation from the connection representation by means of a linear transform called the loop transform[21, 22].

The basic form of the transform is

$$\tilde{\Psi}(\Gamma) = \int d\mu(A) T[\Gamma, A] \Psi(A) \tag{124}$$

where Γ is a loop, or set of loops, γ_i, in which case $T[\Gamma, A] = \prod_i T[\gamma_i, A]$, where $T[\gamma_i, A]$ is the trace of the holonomy of the connection around the loop γ_i in the spin 1/2 representation. Finally, $d\mu(A)$ is a measure on the space of connections.

The motivation for the loop transform mirrors that of the use of the fourier transform in ordinary relativistic quantum theory. For one thing, for simple, non-intersecting differentiable loops, the Wilson loops $T[\Gamma, A]$ are exact solutions of the Hamiltonian constraints. Hence they play the role of plane wave states, in the sense that the loop transform takes components in a basis in which the equations of motion are easily represented. Indeed, the dynamics, whether formulated as a hamiltonian or a hamiltonian constraint, acts at intersections, hence non-intersecting loops are in a sense "free solutions". Furthermore the loop states are invariant under $SO(3)$ gauge transformations so the Gauss's law constraint is automatically incorporated in the transform to the new representation. Finally, as we shall see, the diffeomorphism constraint can be solved in the loop representation, in terms of an infinite dimensional space of solutions, which is not the case in the connection representation.

The spin network states are particular combinations of loop states which are linearly independent and which provide an orthonormal basis for $H^{kinematical}$. Basically they are combinations of loops in which, whenever a number of loops coincide, the products of traces in the fundamental, spin 1/2 representation are decomposed into sums of irreducible representations[23, 24].

An *abstract spin network* is an abstract graph consisting of nodes connected by edges, which is labeled as follows:

Edges are labeled by spins, j, which are the irreducible representations of $SU(2)$.

Nodes are labeled by invariants (also called intertwiners or channels.) If a node has n edges, with labels $j_1, ..., j_n$ the invariant must be a map, $\mu : j_1 \otimes .... \otimes j_n \rightarrow Id$. For each set of representations $j_1, ..., j_n$ is a finite dimensional linear space of such maps, called $V_{j_1,...,j_n}$, the space of intertwiners. So the assignment of an invariant to each node is a choice of a vector in this space.

An *embedded spin network* $\tilde{\Gamma}$ is an embedding of the abstract labeled graph Γ into the spatial manifold Σ. Any embedding is allowed so long as there are no intersections in the image not in the graph. Given an embedded spin network one gets a state $\Psi_{\tilde{\Gamma}}(A)$ in the connection representation by associating to each edge, e, labeled with spin j_e e the parallel transport of A along that edge, in the representation j_e. These are then tied together at each node, n by the invariant μ_n that labels it. The result is a gauge invariant function of the connection, $\Psi_{\tilde{\Gamma}}(A)$, called a spin network state.

The measure $d\mu(A)$ has the property that the spin network states provide an orthonormal basis in which the states $\Psi(A)$ can be expanded. Thus, just like the fourier transform, there is an infinite dimensional Hilbert space $\mathcal{H}$ of functionals which has an inner product given by

$$< \Psi|\chi >= \int d\mu(A)\bar{\Psi}(A)\chi(A). \tag{125}$$

This is called the kinematical inner product.

In the Euclidean theory the connections are real and the inner product and loop transform are defined by an ordinary integral. In the Lorentzian theory the connections are complex, and the loop transform and inner product are defined by a contour integral. The contour is usually taken to be the restriction to real $SU(2)$ values of the connection.

A spin network state can be expanded in terms of loop states, by expanding each representation in terms of symmetric products of fundamental representations. This is, however, rarely the best way to do a calculation. Instead, it is usually best to make use of identities from representation theory.

Since the spin network states make an orthonormal basis, one can then define the transform directly in terms of them by

$$\tilde{\Psi}(\Gamma) = \int d\mu(A) T[\Gamma, A] \Psi(A) \tag{126}$$

where Γ here, and for the remainder of this paper labels an embedded spin network. In both the connection and loop representation it can be shown that the spin network states are linearly independent and are an orthonormal basis of $H^{kinematical}$[21].

These basis elements are labeled $|\Gamma>$. A state in the loop representation is then a functional of spin networks and is labeled,

$$\Psi(\Gamma) =< \Gamma|\Psi > \tag{127}$$

We then have

$$< \Gamma|\Gamma' >= \delta_{\Gamma\Gamma'} \tag{128}$$

That is the inner product is zero unless the two embedded spin networks are identical, in which case it is equal to one. This is clearly very different from the Fock measure. This one should think is good, given the result described in section ?? that there are no exact states which correspond to linearized short wavelength gravitons. At the same time the reader may object that there is a problem, because $H^{kinematical}$ with the inner product (128) is not separable. This is a problem and it is a reason why the loop transform is not useful when applied directly to continuum formulations of gauge theories which are not diffeomorphism invariant. (Although the technique is very useful in the context of lattice gauge theory, where it has been used for a long time, under different names[49]. This problem is however remedied when we go to the subspace of diffeomorphism invariant states, as we shall now see.

12.1 Diffeomorphism invariant states

Once we have transformed to the loop representation, we can study the action of both the diffeomorphism and hamiltonian constraints. The Hamiltonian constraint acts at vertices and generates three or four additional vertices. As this is not the subject of the present paper we refer the reader to [28, 29, 30] where the action of the regulated constraint, and infinite dimensional spaces of solutions, are described in detail for $\Lambda = 0$.

The diffeomorphism constraint can be defined and solved exactly, because the kinematical loop representation carries an exact unitary representation of the diffeomorphism group, defined for a diffeomorphism ϕ, by

$$\hat{U}(\phi) \cdot |\Gamma >= |\phi^{-1} \cdot \Gamma > \tag{129}$$

It is easy to check that this representation is unitary under the inner product (128). A diffeomorphism invariant state $|\Psi>$ is then defined by $\hat{U}(\phi) \cdot |\Psi >= |\Psi >$, which implies that

$$\Psi(\Gamma) = \Psi(\phi^{-1} \cdot \Gamma) \tag{130}$$

[21] Spin network states were introduced in quantum gravity in [24]. But they were known already in the context of lattice gauge theory[91] (although not by that name.) Much earlier Penrose[23] introduced abstract spin networks and argued that they would ultimately provide a discrete theory of quantum geometry, as indeed they have been found to do.

It is easy to solve this and write an infinite number of diffeomorphism invariant states[21]. For example, if $K(\Gamma)$ is any invariant of knots, links and graphs, then

$$\Psi_K(\Gamma) = K(\Gamma) \tag{131}$$

is an exact diffeomorphism invariant state. The space of diffeomorphism invariant states is denoted H^{diff} may then be constructed. It has an orthonormal basis labeled by the diffeomorphism classes of the embedded spin networks[]. If $\{\Gamma\}$ denotes the diffeomorphism equivalence class of the spin network Γ then the elements of this basis are given by

$$\Psi_{\{\Gamma\}}[\Gamma'] = 1 \text{ if } \ \Gamma' \in \{\Gamma\} \text{and } 0 \ \text{ otherwise.} \tag{132}$$

It is not difficult to show that this is a countable basis[22]

For the case of $\Lambda = 0$ an exact, rigorous formulation of the loop transform has been constructed, in terms of a rigorous measure on the space of connections, called the Ashtekar-Lewandowski measure[]. Unfortunately this is not relevant for the case of $\Lambda > 0$, as we shall see. However there is another set of mathematical results that can be used to define the integral transform in this case, coming from studies of Chern-Simons theory.

12.2 The loop transform of the Kodama state

The forgoing discussion of the loop transform and loop representation has been only for $\Lambda = 0$. We now consider the loop transform for $\Lambda \neq 0$. The key point is that for $\Lambda > 0$ we want the state space to include the Kodama state, as we have shown it has all the properties we would require of a vacuum or ground state[23]. The loop transform of the Kodama state is

$$\tilde{\Psi}_K(\Gamma) = \int d\mu(A) T[\Gamma, A] \mathcal{N} e^{\frac{\imath\kappa}{4\pi}\int Y_{CS}} \tag{133}$$

where

$$\epsilon\kappa = k = \frac{6\pi}{G\Lambda}. \tag{134}$$

Up to the possible factor of $\imath$ in κ, this is an expression that is well understood. Indeed, for the Euclidean case, in which $\epsilon = 1$, (133) is nothing but the expectation value, in Chern-Simons theory, of an observable which is the Wilson loop associated to the embedded spin network $\tilde{\Gamma}$. As first shown by Witten's [93] $\tilde{\Psi}_K(\Gamma)$ is a known invariant of knots, links and graphs, which is the Kauffman bracket[94].

The resulting functionals $\tilde{\Psi}_K(\Gamma)$ can be expressed explicitly as a function of $q = e^{\frac{2\pi\imath}{\kappa+2}}$. Thus, for the Lorentzian case the state can be gotten by continuing q to imaginary values of κ.

There is, however, a subtlety in the evaluation of (133), which makes the loop representation somewhat different for $\Lambda \neq 0$. This is that the integrals over the loop are singular, due to the presence of the Chern-Simons term[95].

As a result, the definition of the loop transform of the Kodama state requires a regularization procedure. There are several ways to carry out the regularization, but however it is

[22] This is trivial for graphs with vertices of valence 4 or less, because then the diffeomorphism equivalence classes are countable. For higher valence the argument is more difficult because there are continuous parameters in the labels of the diffeomorphism equivalence classes. The spaces of these parameters are still finite dimensional, so it is still possible to pick a separable inner product. This is discussed in more detail in [92].

[23] The loop transform for $\Lambda \neq 0$ was discussed in [99, 100, 13] and put into the form presented here in [37].

done the result is to introduce additional structure in the definition of the Wilson loops. This structure is called framing, and we must now review the basic ideas involved[93, 96, 95, 97].

The problem is easy to see in perturbation theory. There the path integral must be supplemented with a gauge fixing condition, such as Lorentz gauge, $\partial_\mu A^i_\nu g^{\mu\nu} = 0$. This breaks not only gauge invariance but diffeomorphism invariance, because it requires that we introduce a background metric $g_{\mu\nu}$. The choice of background metric is arbitrary and, as part of a gauge fixing procedure, it must not come into any physical quantities. Once the gauge fixing has been introduced the linearized action may be inverted to find the propagator, $D^{ij}_{\mu\nu}(x,y) \approx 1/|x-y|^2$ However then the leading order diagrams in $< T[\gamma, A] >$ involve a double integral over the loop, of the form

$$< T[\gamma, A] > \approx e^{\int_\gamma ds \int_\gamma dt \dot{\gamma}^\mu(s) \dot{\gamma}^\nu(t) D_{\mu\nu}(\gamma(s), \gamma(t))} + ... \tag{135}$$

This expression has a singularity when $\gamma^\mu(s) = \gamma^\nu(t)$.

The result is that the expression (133) must be regulated. This must be done in such a way that diffeomorphism invariance and gauge invariance are not broken at the end of the calculation. The known ways to accomplish this involve making the loop a two dimensional object, by expanding it in a direction linearly independent of $\dot{\gamma}^\mu(s)$. One way to do this is to attach a small vector $r^\mu(s)$ to the loop at each point such that $r^{[\mu}(s)\dot{\gamma}^{\nu]}(s) \neq 0$. One can then define a family of new loops by $\gamma^\mu_\epsilon(s) = \gamma^\mu(s) + \epsilon r^\mu(s)$ Then (135) is replaced by,

$$< T[\gamma, A] > \approx \lim_{\epsilon \longrightarrow 0} e^{\int_\gamma ds \int_{\gamma_\epsilon} dt \dot{\gamma}^\mu(s) \dot{\gamma}^\nu_\epsilon(t) D_{\mu\nu}(\gamma(s), \gamma_\epsilon(t))} + ... \tag{136}$$

This is finite and well defined. However it turns out to depend on additional information, such as the Gauss linking number of γ and γ_ϵ for small ϵ. This is diffeomorphism invariant, but it is also additional information that must be specified beyond the embedding (up to diffeomorphisms) of the loop γ in the manifold Σ. This additional information goes into the definition of a *framed loop*. A framed loop can be pictured as either a ribbon or a tube embedded in the manifold[97].

It turns out there are better ways to define framed loops and graphs, which relies on a connection to conformal field theory, which was the basis of Witten's original paper [93].

I will not go into the details of this construction here, but here is a heuristic way to think about it[97]. Let us consider first a single loop. Imagine that the loop is blown up to a tube, introducing a boundary to the manifold Σ with the topology of a torus. The torus may be considered ruled, which means that we can identify, up to homotopy, the original loop in it. This is sometimes called a tube "with racing stripes."

The connection on the loop can then be extended to a flat connection on the two dimensional surface of the tube, restricted so that the traced holonomy around any cycle which was also a cycle in the loop has the same value as before, while the holonomy around any cycle which was created by blowing up the loop to a tube is the identity in the group.

The nodes which connected the edges are now blown up to two spheres. The points where the edges attached to the nodes are blown up to circles, which are called punctures. The result is that a closed graph is blown up to a closed 2-surface of some genus. However the 2-surface is ruled, which is to say there is a preferred, up to homotopy, image of the original graph in the two surface.

The connection is extended to a flat connection on the 2-surface, with the requirement that, up to homotopy, any holonomy on a path which is in the image of the original loop is equal to what it was in the original loop.

The analogues of the spin network states will be an orthonormal basis, in some measure, of functions of the flat connections on the 2-surface. However, these have to satisfy some identities, due to the dependence on the framing. For example, by going back to the original integral (133), it can be shown that there are phase factors whenever one of the tubes are twisted by 2π or whenever a punctures on a two sphere representing a node is carried in a circle around one or more other punctures[93, 95, 96]. These phase factors depend on the coupling constant κ that sits in front of the Chern-Simons state.

In conformal field theory the functions on the space of flat connections with these properties are called conformal blocks. One way to notate them is by labeling the tubes by representations, not of the group $SU(2)$, as before, but by representations of the quantum deformed algebra $su_q(2)$ [98]. Here the label q is given in terms of κ, and hence the cosmological constant, by $q = e^{\frac{2\pi\imath}{\kappa+2}}$.

Similarly, the two surfaces, with punctures removed, may be labeled by intertwiners, or invariants, in the representation theory of $su_q(2)$. The structure of the representation theory of $su_q(2)$ turns out to capture exactly the identities required by the regularization of the loop transform of the Chern-Simons state.

We note that the level κ, is in this case an imaginary integer, related to the cosmological constant by eq. by (58). We found in section 11 that k must be integer.

It is interesting to note that for Euclidean quantum gravity, there would be an $\imath$ in the exponential in the Kodama state and κ would be an integer. In this case q is a root of unity. When the deformation parameter $q = e^{\frac{2\pi\imath}{k+2}}$ is a root of unity, with k an integer, the irreducible representations are labeled by half-integral spins, j, except that there is a maximum irreducible representation, given by spin $j_{max} = k - 1$.

The combinatorics of quantum spin networks can also be defined a priori, as has been done by Kauffman and Lins in [98]. One then introduces the notion of an *abstract quantum spin network*, which is an extension of the notion of an abstract spin network defined as follows, given by a deformation parameter q.

- Edges are now tubes, which are ruled as described above. They are labeled by a representation of $su_q(2)$.

- Nodes are now punctured spheres, where each puncture, represented by a little circle on the sphere, is a site by which a tube is attached. These are labeled by intertwiners of $su_Q(2)$. These are also called the conformal blocks, on the punctured sphere, of the conformal field theory called the $su(2)$ WZW model.

- The whole framed graph may be thought of as a ruled two dimensional surface of some genus. The states associated with the surface may also be taken to live in the space of intertwiners, or conformal blocks on that surface. See [97].

An embedded quantum spin network is a quantum spin network embedded in the spatial manifold Σ up to diffeomorphisms of Σ..

Thus, we conclude that to accommodate the Kodama state in the physical Hilbert space, the whole spin-network basis must be quantum deformed to level κ. Thus, Λ *deforms the structure of the Hilbert space of physical states.*

We now note that the loop transform of the Kodama state (133) is also diffeomorphism invariant. Thus, (133) defines an invariant of framed graphs, or equivalently ruled two surfaces, embedded in the spatial manifold Σ. This is called the Kauffman invariant. It is an important invariant of knots and three manifolds.

12.3 Excitations of the Kodama state

We have just defined the loop transform of a single state, the Kodama state. To summarize, in the two representations, we found

$$< \Gamma|\Psi_K >= Kauffman(\Gamma) \rightleftharpoons < A|\Psi_K >= e^{\frac{\imath\kappa}{4\pi}\int Y_{CS}} \tag{137}$$

here Γ is a quantum spin network, which is a ruled two surface as described above, labeled with the representation theory of the quantum group, imbedded in Σ up to diffeomorphisms.

However the theory contains more than one state. In particular, in section 9 we found evidence that there is a large space of solutions to all the constraints which can be gotten by perturbing the Kodama state and which, at least for long wavelength approximate linearized graviton states. We would like to take the loop transform of these states, in order to to express the small perturbations of the Kodama state in a diffeomorphism and gauge invariant language.

Since the cosmological constant is coded, through κ, in the identities that define the quantum spin networks, we may conjecture that a space of the excitations of the Kodama state exists that retains the structure of the quantum group deformation at level κ. Thus, we want to consider a general state of the form,

$$\Psi = \sum_\Gamma c(\Gamma)|\Gamma > \tag{138}$$

where Γ, which labels the basis states, are quantum spin networks, as defined above, imbedded up to diffeomorphisms in Σ.

There is a natural inner product that can be defined on the space of states (138), which is defined as follows[37]. Let us consider a two surface Δ embedded up to diffeomorphisms in Σ. This two surface has a space of flat $SU(2)$ connections $\mathcal{V}_\Delta$. This space is finite dimensional and compact. The functionals on $\mathcal{V}_\Delta$ are called the conformal blocks. They may be labeled by the intertwiners of the $su_q(2)$ algebra. If we neglect the ruling, then there is a natural inner product on $\mathcal{V}_\Delta$ given by the representation theory of $su_q(2)$. In this inner product, different states associated with different rulings of the two surface are not generally orthogonal. Instead, they are connected by identities which extend the identities of the representation theory of $SU(2)$ which give the definition of the $6j$ symbols.

To denote the inner product, let us denote a quantum spin network by the pair $\Gamma = (\Delta, \phi)$, where Δ is the embedding of a two surface up to diffeomorphisms in Σ and ϕ is a state in $\mathcal{V}_\Delta$. Then we can define the inner product by

$$< \Gamma|\Gamma' >= \delta_{\Delta\Delta'} < \phi|\phi' >_{\mathcal{V}_\Delta} \tag{139}$$

Thus, two quantum spin network states are always orthogonal if they live on two surfaces Δ and Δ' of different genus, or if their two embeddings are non-diffeomorphic. However, if the two surfaces are diffeomorphic their inner product is given by the natural inner product in $\mathcal{V}_\Delta$.

The normalizable Ψ of the form (138) with inner product (139) span a Hilbert space $\mathcal{H}^{qdiffeo}$. This is an infinite dimensional state space, which is an extension of the space of diffeomorphism invariant spin network states.

We may note that in this inner produce two quantum spin network states which differ by an extension of the recoupling identity, extended to the quantum group, represent the same state. This contracts somewhat the space of states compared with the ordinary spin network states, where there is no such identity. At the same time, the space of states is still infinite dimensional, because two states associated with quantum spin networks with surfaces Δ of different genus will be orthogonal. Since the genus can be an arbitrary non-negative integer this means the Hilbert space is infinite dimensional.

A small perturbation of the Kodama state will then have the form,

$$c(\Gamma) = Kauffman(\Gamma) + k^{-1}\delta c \tag{140}$$

We may note that this automatically resolves a difficulty with the theory at $\Lambda = 0$, having to do with the dynamics generated by the Hamiltonian constraint. Briefly, here is the essence of the problem[70, 71] and its solution[37].

In either the hamiltonian or path integral (spin foam) formulation of loop quantum gravity one finds that the dynamics consists of local changes in the spin networks. An example of such a local move is the replacement of one trivalent vertex by a triangle containing three new vertices. This is called the $1 \rightarrow 3$ move[36]. By hermiticity, the time reverse of this move, denoted $3 \rightarrow 1$ move must also be present.

In fact the $1 \rightarrow 3$ and $3 \rightarrow 1$ moves are all that are generated by certain forms of the Hamiltonian constraint[28, 29, 30]. These arise from a point splitting regularization technique where the different operators that make up the hamiltonian constraint are split apart in the spatial hypersurface Σ.

However, there is another kind of move that can be defined on spin networks, which is called the $2 \rightarrow 2$ move. This acts on a pair of nodes that share a common edge, swapping their inputs, as in the usual angular momentum recoupling identities.

The key point is that the theory must contain these $2 \rightarrow 2$ moves if general relativity is to be recovered as the low energy limit. One reason is that only when these moves are included can the dynamics satisfy the property that any finite spin network can evolve to any other one in a finite number of moves. A dynamics generated only by the $1 \rightarrow 3$ and $3 \rightarrow 1$ moves divides the Hilbert space of spin network states into an infinite number of sectors that do not mix under the dynamics. This is inconsistent with what is expected from the Einstein equations, for it implies that there are an infinite number of observables, or constants of the motion, that measure simple, quasi local properties of the states, but which commute with the Hamiltonian constraint. This would, at best, imply that the theory was integrable, which we certainly do not expect to be the case. But even more simply, one can show that in the absence of $2 \rightarrow 2$ moves information is not propagated in the quantum spacetimes, because it can be shown that information at one node of an initial spin network never propagates even to nearby nodes.

There are still other difficulties with the forms of the Hamiltonian constraint found in [] that neglect the $2 \rightarrow 2$ moves. These involve problems with the algebra of constraints[] as well as problems with preserving positive energy in the quantum theory[].

The inclusion of the $2 \rightarrow 2$ moves solves many, if not all, of these problems. It is also possible to see that these moves are required by spacetime relativistic invariance, and that the problem arises from the fact that the regularization procedures used in the construction of the operators impose a preferred spatial slicing in which the point splitting is done.

It is then very interesting to notice that the $2 \rightarrow 2$ moves are automatically included once the theory is expressed in terms of quantum deformed spin networks. The reason is that when $q \neq 1$ the states that differ by these moves are no longer orthogonal to each other in the inner product just defined[37]. Thus, any state has a non-vanishing amplitude to be found in a state that differs by an application of one or more $2 \rightarrow 2$ moves.

13 The inclusion of spacetime boundaries

Up till this point we discussed the quantum theory for a spatial manifold with compact topology. We now consider the case in which the region of spacetime which is treated quantum mechanically has a boundary.

In section 6 we discussed the general problem of how to deal with boundaries in the classical theory, after which we studied a class of boundary conditions called the "Chern-Simons boundary conditions", eq. (56).

These boundary conditions can be quantized in the Hamiltonian approach and the physical picture that results is the following[24]

Note that when eq. (56) is imposed, neither the metric nor the connection is fixed on the spatial boundary. So the boundary is, like Σ, a manifold without metric.

The quantum spin networks can end on the spatial boundary. They end in punctures, labeled by reps of $SU_q(2)$, which we denote $\{j\}$.

To understand the physics of the boundary we can use the fact that the operator which measures the area of the boundary is well defined. Its eigenvalues are associated with different sets of punctures, and have the form,

$$A[\mathcal{B}] = \frac{\hbar G}{2} \sum_j \sqrt{C_j} = \frac{\hbar G}{2} \sum_j \sqrt{j(j+1)} + O(1/k) \tag{141}$$

where C_j is the quadratic Casimer of the quantum group $su_q(2)$.

Thus, for an eigenstate of $A[\mathcal{B}]$ the boundary, $\mathcal{B}$, can be represented as a compact surface with a fixed set of punctures.

For each choice of punctures, the Hilbert space decomposes into a product of a bulk piece and a boundary piece. Thus, the whole hilbert space in the presence of the boundary has the form,

$$H = \sum_{\{j\}} H^{bulk}_{\{j\}} \times H^{boundary}_{\{j\}} \tag{142}$$

$H^{bulk}_{\{j\}}$ has a basis given by quantum spinnets that end on the punctures $\{j\}$.

To understand the boundary Hilbert space, note that the dual of the frame field, pulled back into the boundary, $E_{ab} = \epsilon_{abc} E^c$ (which is also the self-dual two form of the metric, pulled back into $\mathcal{B}$) is fixed by (56) to be proportional to the curvature. Hence the only degree of freedom which survives on the boundary are the components of the connection, A_a, pulled back to the boundary.

The next thing to notice is that there is a time derivative in the boundary term, as it is after all the action of the three dimensional Chern-Simons theory. The Poisson brackets

[24] The Chern-Simons boundary conditions were introduced in [13] first for the Euclidean case. Kirill Krasnov realized they apply also to the horizon of a black hole[14], this led to the development of the *isolated horizon* approach to the quantum geometry of horizons[15]. The application of Chern-Simons boundary conditions to time like boundaries with $\Lambda \neq 0$ was developed in [16] and applied, for $\Lambda < 0$ to supergravity in [17].

and hence the commutation relations are then altered on the boundary. On the boundary the canonical commutation relations from Chern-Simons theory hold[13],

$$[A_a(\sigma), A_b(\sigma')] = \frac{2\pi\hbar}{k}\frac{\imath}{\epsilon}\epsilon_{ab}\delta^2(\sigma,\sigma') \tag{143}$$

where we recall that $\epsilon = \imath$ for the Lorentzian theory and $\epsilon = 1$ for the Euclidean continuation. Thus, in either case the boundary Hilbert space is related to the Hilbert space of Chern-Simons theory, with $\kappa = \epsilon k$. To see how we consider the effect of the boundary condition.

The action of the boundary condition (56) can best be understood by integrating it against a test function $g(\sigma)$ on $\mathcal{B}$. They then have the form

$$\int_{\mathcal{B}} d^2\sigma^{ab} F_{ab}\ g(\sigma) = \frac{\Lambda}{3}\int_{\mathcal{B}} d^2\sigma^{ab}\epsilon_{abc}E^c\ g(\sigma). \tag{144}$$

The operator $E^c(\sigma)$ will only be non-zero if it acts at a point where an edge of the quantum spin network meets the boundary. So we deduce that for all regions in which there is no edge attached, the curvature F_{ab} vanishes.

When there is an edge attached in the region where g has support the effect is the same as a delta function singularity of the form,

$$F^i_{ab}(\sigma) \approx \frac{2\pi}{k} X^i_j \epsilon_{ab}\delta^2(\sigma,\sigma') \tag{145}$$

where j is the representation labeling the edge at a puncture at the point σ' and X^i_j is an element of the lie algebra in the conjugacy class of j.

Thus, for a fixed set of punctures, $\{j\}$, the boundary Hilbert space is the Hilbert space of Chern-Simons theory, at level κ, on the two surface, which is the boundary with the fixed set of punctures. This is called $\mathcal{V}_{\{j\}}$. The boundary condition has just fixed the punctures of the boundary to be points at which ends of the quantum spin nets are attached.

The Hilbert space of Chern-Simons theory on a punctured surface, with (145) imposed is well understood[97, 93, 96] for the case of real level, which is the case of the Euclidean continuation. It has several characterizations. First, it is isomorphic to the space of intertwiners of the quantum group, for the products of the representations labeling the punctures. It is also the space of conformal blocks of the $SU(2)$ Wess-Zumino-Witten theory on the punctured surface.

Given that the state we have identified as the ground state of the theory satisfies the KMS condition to be a thermal state, it makes sense to seek to derive thermodynamic relations concerning the quantum theory. When we do this we should use the Euclidean signature version of the theory, as it is the Euclidean histories that are integrated over in the path integral representation of a thermal state. Thus, in what follows we take $\kappa = k = 6\pi/\Lambda$ to be an integer.

13.1 Automatic satisfaction of the Bekenstein bound

The boundary hilbert space we have defined is finite dimensional for a finite number of punctures. Moreover the Bekenstein bound[101] is automatically satisfied, as we now show[13].

For a fixed set of punctures, $\{j\}$ the Hilbert space is an eigenspace of the operator that measures the area of the boundary, with eigenvalue given by eq. (141). Thus, the eigenspaces of the area operator comprise the finite dimensional vector spaces, $\mathcal{V}_{\{j\}}$. The exact value of the dimension of $\mathcal{V}_{\{j\}}$ is given by the Verlinde formula[102]. However it is easy to estimate for a large number of punctures, and large k. In this case we may approximate its dimension by the dimension of the space of invariants in the product of the representations j. For large k the quantum dimensions of these representations are close to the dimensions of the representations of the classical lie algebra. Neglecting a small factor this gives

$$dim\mathcal{V}_{\{j\}} = \prod_j (2j+1) \tag{146}$$

The Bekenstein bound is the statement that

$$\ln dim\mathcal{H}^{bound} < \frac{A[\mathcal{B}]}{4G_R\hbar} \tag{147}$$

where G_R is the renormalized value of Newton's constant, which is the one measured in the classical theory. This will be related to the G of the fundamental theory by

$$G_R = cG \tag{148}$$

where c is a multiplicative renormalization constant. Since the theory has an ultraviolet cutoff we expect c to be finite, in fact c is of order unity, as we now see.

By comparing the expressions for area and the dimension of the boundary Hilbert space, we see that the Bekenstein bound will always be satisfied, for large k and large area, so long as c can be chosen such that it is always the case that

$$\sum \ln(2j+1) < \frac{c}{4}\sum_j \sqrt{j(j+1)} \tag{149}$$

This is the case, by inspection, and we see that the constant is fixed by the smallest representation, $j = 1/2$. So we find, $c = \frac{ln(2)}{\sqrt{3}}$.

So we learn that the Bekenstein bound is satisfied up to a constant of order one, in the bare newton's constant. If we require that it hold exactly with the 1/4 then we fix the renormalization of the gravitational constant.

13.2 Applications of the boundary theory

As mentioned in section 6, there are several cases in which the boundary theory is of this general form, depending on whether the spacetime boundary is null or timelike. There is a large literature about the case where the boundary is an horizon, satisfying certain conditions, which amount physically to the condition that the spacetime is stationary in the neighborhood of the horizon. Using the appropriate specialization of the Chern-Simons boundary conditions the black hole entropy may be derived and, in fact, explained, as the log of the number of quantum states at the boundary which are compatible with the condition that the boundary is, locally, an horizon[14, 15]. There are further results which show that, at least heuristically, the Hawking radiation can be derived[39]. Most interestingly, there are calculations that indicate that the Hawking entropy has a fine structure and that there are corrections to the relationship between area and entropy[41, 39, 40].

Another case that has been studied is that of time like boundary conditions, for the case of negative Λ [16, 17]. There are indications here that something like the AdS/CFT correspondence may exist in $3+1$ dimensions, when the limit is taken in which the area of the spatial boundary goes to infinity[103].

There has been less work on the imposition of boundaries when the cosmological constant is positive, but some of the results of [16, 17] apply to this case as well. I will describe here just the main features the theory, details will appear elsewhere[103].

It is important to stress that for the interpretation of the constant c as a finite renormalization of Newton's constant to hold, it must be universal, and all calculations must lead to a single value for c. This is so far the case in general relativity, whatever type of boundary or horizon has been studied, c turns out to be universal.

However, c does differ in supergravity, where it depends on the number of supersymmetry generators[17]. Hence if there were no other way to detect supersymmetry, it could in principle be detected by measuring the quanta of area in units of the observed, G_R.

13.3 The N-bound

The boundary theory developed here gives some insight into an important conjecture of Banks, called the N bound[18]. Motivated by the finiteness of the entropy of a horizon in deSitter spacetime, Banks conjectured that a quantum theory of gravity will have only a finite number, N, of degrees of freedom when λ is positive, with

$$N = \frac{3\pi}{\Lambda} \tag{150}$$

For a semiclassical quantum field theory in deSitter spacetime, the N bound follows from Bousso's form of the holographic bound[19]. Bousso further argues that the bound holds for any semiclassical theory in a spacetime with $\Lambda > 0$, at least so long as certain kinds of matter fields are excluded[19].

Here is the basic semiclassical argument for the N bound. In deSitter spacetime the horizon of any observer has area,

$$A_{max} = \frac{12\pi}{\Lambda} \tag{151}$$

This is hence the largest surface from which an observer in deSitter spacetime could receive information. If we believe in the Bekenstein bound then we would say that the most information that could be read by an observer on the horizon is N_{max} where

$$N_{max} = \frac{A_{max}}{4} = \frac{3\pi}{\lambda} \tag{152}$$

This is hence the maximum dimension of a Hilbert space needed to represent the information that an observer in deSitter space could measure.

To generalize beyond deSitter spacetime, one may make use of results that suggest that, when $\Lambda > 0$, any spacetime satisfying the usual energy conditions will asymptotically approach deSitter spacetime. This is because, for the usual forms of matter, in time the exponential expansion dilutes the effect of matter and gravitational radiation, so that the effective energy momentum tensor is dominated by the cosmological term.

However, if the N bound is really fundamental, it should apply not to the degrees of freedom of a semiclassical quantum field theory, in a fixed classical spacetime background, but to the full quantum theory of gravity. Thus, it is interesting to see if the bound may be derived directly from the full quantum theory of gravity in the case $\Lambda > 0$.

We consider the case that k is large, but finite, so that Λ is small in Planck units. We consider also the case that the universe is described by a thermal state close to the Kodama state. In this case the universe can be thought of as close to thermal equilibrium, which corresponds to the fact that classically deSitter spacetime is stationary. We can then use the Euclidean formalism to describe what an observer measures.

Since k is very large, the horizon is far from the observer and the observer receives information from quanta which are radiated from the horizon to the observer.

We next note that the gauge group $SO(3)$ corresponds to the freedom of the observer to rotate locally. It then follows that *the information received by the observer at any one time should*[25] *be contained in an irreducible representation of* $SO(3)$.

However, we have seen that when $\Lambda \neq 0$ the representation theory of the local rotation group is quantum deformed. This means that there are non-trivial quantum effects, due to the presence of the cosmological constant, which modify what happens to the observer's view of the universe when he rotates. So the preceding statement should be modified to say that *the information received by an observer at any one time should be contained in an irreducible representation of* $SU_q(2)$.

However, when k is a finite integer there is a largest representation of $SU_q(2)$. As a result there is a limitation on the amount of information that could be measured by a local observer in at any given time. The bound on the information measurable corresponds to the dimension of the largest irreducible representation of $SU_q(2)$.

Now, the largest representation with k finite is[98]

$$j_{max} = \frac{k}{2} = \frac{3\pi}{\lambda} \tag{153}$$

But j_{max} is also the number of degrees of freedom. This is because, for a given area, the dimension of the space of intertwiners, or invariants, is extremized when the spins have the lowest possible value. Thus, maximal entropy screens have some number, n, of spin 1/2 punctures. The entropy such a screen is then given for large n and large k approximately by

$$S(n) \approx \ln(dim[\mathcal{H}_{1/2}])n = n\ln(2). \tag{154}$$

The corresponding number of degrees of freedom is

$$N = \frac{S(n)}{\ln(2)} = n \tag{155}$$

because entropy is counted in units of Shannon information.

But $N_{max} = \frac{k}{2}$, so we have

$$N_{max} = \frac{3\pi}{\Lambda} \tag{156}$$

This is the N-bound.

To summarize, we have given a physical argument in support of a connection between causality, measurement and irreducibility, in the context of loop quantum gravity. The connection is that the information received by any local observer at any one time must be contained in some irreducible representation of the local gauge group. Otherwise, there are quantum states, observable by a local observer in spacetime, that do not transform irreducibly when that observer rotates locally. We have then shown that this conjecture implies the N-bound.

[25] This requirement brings to mind the origin of the word *universe* as "that which turns as one."

13.4 How to do quantum cosmology with horizons

Before closing, I want to make a few remarks concerning some of the deeper problems the quantum theory of gravity faces, concerning the application of the quantum theory to cosmological spacetimes in which the observers, along with their measuring instruments and clocks, must be considered part of the quantum system. While this problem remains open, I would like to describe here a new approach which has been proposed and developed by people working in loop quantum gravity, which has come to be called *relational quantum cosmology.*

The problem of quantum cosmology is especially acute for the case $\Lambda > 0$ as classical spacetimes with $\Lambda > 0$ generically have horizons. This means that no real observer inside the universe can observe the whole universe. In such circumstances there is a limit to much of the universe any observer can see, no matter how long they wait.

To proceed we may distinguish three closely related issues:

-What is the right formal structure for quantum theory in a cosmological context? Should it be based on the conventional Hilbert space and algebra of observables, or on something more exotic?

-What is the right measurement theory for quantum cosmology? How do we deduce predictions for real experiments from the mathematical states and operators, or whatever replaces them? Here it must be stressed that if the theory is going to be compared with observations we can make, the measurement theory cannot just produce predictions for asymptotic observers, at very late times. To be useful for us the measurement theory must make sense of observers inside the universe, at times which, however large they may be on Planck scales, are still short compared to the lifetime of the universe, or the time it takes the universe to reach some final asymptotic state.

Thus, whether or not there are ultimately horizons, it is still the case that finite observers such as ourselves only have access to a small fraction of the whole universe. Thus, whatever the value of the cosmological constant the measurement theory must allow a situation where the only physically possible observations are partial observations.

-However, there may still be special features of a quantum cosmology with a positive cosmological constant, which the measurement theory must incorporate.

Over the last fifty or so years there have been several serious attempts to solve the problem of quantum cosmology. These attempts are of two kinds. There are approaches to quantum cosmology that take the mathematical structure of the theory to be the same as that of ordinary quantum mechanics which we may call "conventional quantum cosmology". All the other approaches propose that the formal structure of quantum theory must be modified for the theory to be sensibly applied to cosmology.

Quantum cosmology is a controversial subject.

The problem is, briefly, that the major interpretations of quantum mechanics rely on a division of the world into a classical part, which contains the observer, her clocks and measuring instruments and a quantum part, which includes the system under study.

The Copenhagen interpretations of quantum theory, as propounded, in somewhat different forms by Bohr, Heisenberg, Dirac, von Neumann and others all explicitly introduce this split. Since then other interpretations of quantum mechanics have been proposed, such as the Everett interpretation, the many universes interpretation, the consistent histories interpretation, etc, that do not rely on a split of the world into a classical and a quantum part. Thus, these interpretations seem to allow the whole universe to be described as a single quantum system. However, to make a very long story short, when developed in practice, each of these interpretations relies on an additional ad hoc hypothesis, which is introduced to make a connection between calculations in quantum theory and real observations made by real observers. While this is not the subject of this paper, I believe it is the case that none of these interpretations really solved the problem of how to apply quantum theory in a cosmological situation.

As a result, over the last several years a new approach has been proposed[56]-[60]. The main idea of these approaches is to take seriously Bohr's idea that the measurement theory of any quantum theory must be based on a split of the universe into two parts, the quantum system, which is modeled in a hilbert space, and a classical part, which includes the observer along with her clocks and measuring instruments. Bohr always stressed that the division between these two worlds must be arbitrary, so that the physical predictions of the theory do not depend on how it is made. Bohr also stressed that the quantum state was not so much an objective property of the quantum system as it was something dependent on both sides of the split, as it contained the information needed to deduce what will be the result of all of the various measurements accessible to the classical observer, carried out on the quantum system.

These ideas of Bohr are among the most mysterious features of his thought, for it appears to imply that the quantum state, and the Hilbert space in which it lives, are not completely objective properties of the quantum system, as they depend to some extent on how the quantum/classical split is made.

The main idea of relational quantum cosmology is to take Bohr seriously, by following the methodology of special relativity. Rather than looking for a preferred observer, or a preferred split of the world into its quantum and classical parts, the idea is to include in the formalism of the quantum theory all possible ways of dividing the universe into two parts. There are then many Hilbert spaces, one for each possible way to divide the universe into a quantum and classical part. Each Hilbert space becomes an objective property, but not of the system, but of the boundary between the classical and quantum world.

The basic ideas of relational quantum cosmology may be summarized as follows:

- Crane: *Hilbert spaces are associated with boundaries that split the universe into parts. By the relationship of GR to TQFT these will be described in terms of finite dimensional state spaces. Hence the Bekenstein bound.*[56]

- Rovelli: *Each hilbert space describes the information one part of the universe has about another part. The various measurement problems are solved by paying attention to the fact that different observers record their measurements in different Hilbert spaces.* [57]

- Dyson: *For each observer, the description of the past is classical, while the present and future are described quantum mechanically.*[104]

- Markopoulou: *The natural way to divide the world into quantum quantum systems and observers is to use the causal structure of spacetime. In a cosmology there is then a Hilbert space for each local region of spacetime, in which is represented the information that arrives there from its causal past. These are tied together by maps which incorporate the dynamics and express the causal relations*[26].[58]

[26]These structures are mirrored also in the algebra of observables of any classical, relativistic cosmological

- Butterfield and Isham: *The right mathematics for relational quantum theory is topos theory and more particularly, the structure of the Hilbert spaces for the quantum theory of spacetime is that of a presheaf over a partially ordered set.* [105]

Reduced to a slogan, relational quantum cosmology maintains that, "Many quantum states to describe one universe, not one state describing many universes."

From this brief description, it is clear that relational quantum theory is closely related to the Holographic Principle, according to which a quantum gravitational system may be described in terms of information measurable on its boundary. Indeed, the earliest, to my knowledge, statement of the holographic principle, is in a paper by Louis Crane in which he formulated also the basic idea of relational quantum theory. That paper was, in turn, inspired by the relationship between quantum gravity and topological quantum field theory.

A version of the holographic principle compatible with the ideas of relational quantum cosmology was then proposed in [60][27]. it is called the *weak holographic principle* and may be summarized as follows,

- A quantum spacetime has a discrete causal structure, in which the events are transitions, or local moves, in a (quantum) spin network. The analogue of spacelike slices are (quantum) spin networks[28].
- The causal structure extends to spacelike surfaces, which are defined in terms of the topological dual of the quantum spin networks. Thus, to each spacelike two surface is associated a set of representations $\{j\}$ which are the labels of the edges it is dual to, or intersects.
- Any spacelike surface in a quantum spacetime can be considered a channel through which quantum information flows from its causal past to its causal future. To each such spacelike two surface is associated the finite dimensional Hilbert space $\mathcal{V}_{\{j\}}$. The information flowing through the surface is represented by an operator in $\mathcal{V}_{\{j\}}$.
- All measurements are made on such surfaces, hence all observables in a quantum spacetime are operators in some $\mathcal{V}_{\{j\}}$.
- The area of the surface is another name for its capacity as a channel of quantum information. Thus, the Bekenstein bound, which as we have shown is automatically satisfied, is interpreted as a reduction of area to a measure of information flow.

If these ideas are right, they should apply to quantum gravity with a cosmological constant. Each observer in deSitter spacetime is able to receive information from only that part of the spacetime interior to her horizon. We indeed found that we could develop the semiclassical theory in the interior of each such region, characterized by a flat slicing of a region of deSitter spacetime. In the full quantum theory the horizon may be represented by a boundary, on which the self-dual boundary conditions discussed here may be applied.

theory which is not Boolean but Heyting. This is due to the the fact that no observer can give truth values to all propositions about the history of the universe. This algebra has no representation in terms of projection operators in a single hilbert space.

[27] Its relationship with other proposed form of the holographic principle is discussed in [61].

[28] The whole $3+1$ dimensional structure is called a causal spin foam, and is described in [36, 37, 106].

14 Conclusions and further developments

In this paper we have presented a number of results, some old and some new, which together support a claim that for $\lambda > 0$ and in $3+1$ dimensions loop quantum gravity gives a satisfactory quantum theory of gravity. The main results were stated in the introduction.

There are a number of related developments which I did not have space to mention here. These include

- The idea that gravity is a constrained topological field theory turns out to be very powerful when applied to the derivation of measures for the path integral formulations of quantum gravity, which are called spin foam models[32, 34, 35].
- A related model, called a causal spin foam model, has been developed for $\Lambda > 0$[37].
- The Chern-Simons state has been studied in the context of reduced quantum cosmological models, in which only a few degrees of freedom are retained, such as in the quantum Bianchi models[107].
- The Chern-Simons state can be transformed to the triad, or frame field representation[108].
- Soo has described an expansion around the Kodama state in powers of λ [8].
- A strong coupling expansion, in powers of $\frac{1}{\lambda}$ was proposed in [28] and has been developed in some detail in [109].
- The Kodama state can be used to generate an infinite number of physical states at $\lambda = 0$[110].
- There are related results also for $\Lambda < 0$, including a number that extend to supergravity for $N = 1$ and $N = 2$ [17, 51, 50].

There remain several things that must be done to clinch the case. These include, 1) development of the perturbation theory for the excitations of the Kodama state, leading to computations of corrections to graviton-graviton and graviton-matter scattering. This will require an understanding of how to expand small excitations of the Chern-Simons state in the basis of exactly gauge and diffeomorphism invariant states. 2) More study of the corrections to the energy momentum relations[29], including calculations for photons and protons and an understanding of the consequences for either breaking or modifications of lorentz invariance at Planck scales. 3) Development of the relational framework for quantum cosmology, to give a complete theory of quantum spacetimes with $\Lambda > 0$.

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[29]This is in progress with Carlo Rovelli and additional results will be reported in [82].

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